University of Thi-Qar Journal Vol.13 No.1 Mar 2018
Web Site: https://jutq.utq.edu.iq/index.php/main
Email: journal@jutq.utq.edu.iq

On Fuzzy Soft ideal topological Spaces

Amneh kareem Yousif, Hasan Ali Naser

https://doi.org/10.32792/utq/utj/vol13/1/1

Department of Mathematics, College of Education for Pure Sciences, University of Thi qar.
Amneh.k@utq.edu.iq, Haalna28@utq.edu.iq
Azhar Aziz Sangoor
Department of Mathematics, College of Computer Sciences and Mathematics
Azharaziz1984@gmail.com

Abstract:

In this work, we generalized the concepts of fuzzy ideal and r-fuzzy local function in fuzzy ideal topological space in view of the definition of Šostak to fuzzy soft topological spaces. Also, we proved some results and connections among them.

الخلاصة:

في هذا البحث قمنا بتعميم مفاهيم المثالية الضبابية والدالة r –الضبابية المحلية في الفضاء التبولوجي الضبابي المثالي حسب تعريف Šostak للفضاء التبولوجي الضبابي , الى الفضاءات التبولوجية الضبابية الواهنة. كذلك برهنا بعض النتائج الخاصة بهذه التعميمات والعلاقات بينها.

---

1. Introduction:

The concept of fuzzy set introduced by Zadeh [14] in 1965. In 1999, Molodtsov [6] introduced the notion of soft set and started to develop the basic of the corresponding theory as a new approach for modeling uncertainties. Maji et al [4] defined the concept of fuzzy soft sets. Roy and Samanta [7]did some modification in fuzzy soft set analogously ideas made for soft set. In 1968 Chang [2] defined the fuzzy topology as a family of fuzzy sets satisfying three classical axioms. Later fuzzy topology generalized in different ways, one of these ways is developed by Šostak [11]. After that several authors extended various concepts in classical topology to the Šostak's fuzzy topology. Shabir and Naz [10] introduced the concept of soft topological space. Tany and Kandemir [12] introduced the definition of fuzzy soft topology over a subset of the initial universe set. Later many authors studied it as Roy and Samanta [7], Varol and Aygun [13] and Cetkin [1] etc. kalpana and Kalaivani [3] introduced the concept of fuzzy soft topological space in Šostak sense. Saber and Abdel Sattar [8] introduced the new type of fuzzy ideals and fuzzy local function namely fuzzy ideal and r-fuzzy open local function and studied many of characterizations, properties and connections between it and other corresponding fuzzy concepts are studied.

In this paper, we introduced the notions of fuzzy soft ideal and r-fuzzy soft open local function and we proved some results related them and discussed some relations between them.

2. Preliminaries

Throughout this paper, let $I = [0,1]$, $I_0 = (0,1]$.

Definition 2.1 [14]

A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership " of x in A for $x \in X$.

Let I^X denotes the family of all fuzzy sets on X. If $A, B \in I^X$, then some basic set operation for fuzzy sets are given by Zadeh as follows:

(1) $A \le B \iff \mu_A(x) \le \mu_B(x)$, for all $x \in X$.
(2) $A = B \iff \mu_A(x) = \mu_B(x)$, for all $x \in X$.
(3) $C = A \vee B \iff \mu_C(x) = \mu_A(x) \vee \mu_B(x)$, for all $x \in X$.
(4) $D = A \wedge B \iff \mu_D(x) = \mu_A(x) \wedge \mu_B(x)$, for all $x \in X$.
(5) $E = A^c \iff \mu_E(x) = 1 - \mu_A(x)$, for all $x \in X$.

A fuzzy point in X, whose value is α $(0 < \alpha \le 1)$ at the support $x \in X$, is denoted by x_α. A fuzzy point $x_\alpha \in A$, where A is a fuzzy set in X iff $\alpha \le \mu_A(x)$. The class all fuzzy points will be denoted by $S(X)$.

Definition 2.2 [5]

For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident with B, i.e., there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If A is not quasi-coincident with B, then we write $A\overline{q}B$.

Definition 2.3 [6]

Let X be the initial universe set and E be the set of parameters. A pair (F, A) is called a soft set over X where F is a mapping given by $F: A \to P(X)$ and $A \subseteq E$.

Definition 2.4 [6]

Let $(F, A)$and (G, A) are two fuzzy soft sets in X, then

(1) $(F, A) \subseteq (G, A)$ if $F(e) \subseteq$ G(e) for every $e \in E$.
(2) $(F, A) = (G, A)$ if $(F, A) \subseteq (G, A)$ and $(G, A) \subseteq (F, A)$.
(3) (F, A) is said to be null soft set if for every $e \in E$, $F(e) = \emptyset$.
(4) (F, A) is said to be absolute soft set if for every $e \in E$, $F(e) = X$.
(5) If $\{(F_i, A): i \in I\}$ be a family of soft sets in X, then
$\bigcup_{i\in I}(F_i, A) = \bigcup \{F_i(e): i \in I\}$ for every $e \in E$.
(6) If $\{(F_i, A): i \in I\}$ be a family of soft sets in X, then
$\bigcap_{i\in I}(F_i, A) = \bigcap\{F_i(e): i \in I\}$ for every $e \in E$.

Definition 2.5 [7]

Let $A \subseteq E$. A fuzzy soft set f_A over universe X is a mapping from the parameter set E to I^X, i.e., $f_A: E \to I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subset E$ and $f_A(e) = 0_X$ if $e \notin A$ where 0_X denotes empty fuzzy set on X.

The family of all fuzzy soft sets in X denoted by $\widetilde{(X, E)}$.

Definition 2.6 [7]

Let $f_A, g_B \in \widetilde{(X, E)}$ and $A \subseteq E$, then

(1) f_A is called a fuzzy soft subset of g_B if $f_A(e) \leq g_B(e)$ for every $e \in E$ and we write $f_A \sqsubseteq g_B$.
(2) f_A and g_B are said to be equal, denoted by $f_A = g_B$ if $f_A \sqsubseteq g_B$ and $g_A \sqsubseteq f_B$.

(3) The union of f_A and g_B is also a fuzzy soft set denoted by h_C, defined by $h_C(e) = f_A(e) \vee g_B(e)$ for all $e \in E$, where $C = A \cup B$. Here we write $h_C = f_A \sqcup g_B$.

(4) The intersection f_A and g_B is also a fuzzy soft set denoted by h_C, defined by $h_C(e) = f_A(e) \wedge g_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \sqcap g_B$.

(5) The fuzzy soft set f_A is called null fuzzy soft set if for all $e \in E$, $f_A(e) = 0_X$, dented by 0_E.

(6) The fuzzy soft set f_A is called universal fuzzy soft set if for all $e \in E$, $f_A(e) = 1_X$, dented by 1_E.

Definition 2.7 [12]

Let $f_A \in \widetilde{(X, E)}$. The complement of f_A denoted by f_A^c, is a fuzzy soft set defined by $f_A^c = 1 - f_A$ for every $e \in E$.
Let us call f_A^c to be fuzzy soft complement function of f_A. Clearly $(1_E)^c = 0_E$ and $(0_E)^c = 1_E$.

Definition 2.8 [9]

The fuzzy soft set $f_A \in \widetilde{(X, E)}$ is called fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in X i.e. there exist $x \in X$ such that $f_A(e)(x) = \alpha$ $(0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denoted this fuzzy soft point $f_A = e_x^\alpha = \{(e, x_\alpha)\}$.

The set of all fuzzy soft points in X denoted by $FSP\ (X)$.

Definition 2.9 [9]

Let $f_A, e_x^\alpha \in \widetilde{(X, E)}$, read as e_x^α belongs to the fuzzy soft set f_A if for the element $e \in A, \alpha \leq f_A(e)(x)$.

Definition 2.10 [9]

Let $f_A, g_B \in \widetilde{(X, E)}$. f_A is said to be soft quasi-conicident with g_B denoted by $f_A q g_B$. if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$. If f_A is not soft quasi-conicident with g_B, then we write $f_A \overline{q} g_B$.

Lemma 2.11 [3]

Let Δ be an index set and $f_A, g_B, h_C, (f_A)_i, (g_B)_i \in \widetilde{(X, E)}, i \in \Delta$, then we have the following properties:

(1) $f_A \sqcap f_A = f_A$, $f_A \sqcup f_A = f_A$.

(2) $f_A \sqcap g_B = g_B \sqcap f_A$, $f_A \sqcup g_B = g_B \sqcup f_A$.
(3) $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C$, $f_A \sqcap (g_B \sqcap h_C) = (f_A \sqcap g_B) \sqcap h_C$.
(4) $f_A \sqcap (\sqcup_{i\in\Delta} (g_B)_i) = \sqcup_{i\in\Delta} (f_A \sqcap (g_B)_i)$.
(5) $f_A \sqcup (\sqcap_{i\in\Delta} (g_B)_i) = \sqcap_{i\in\Delta} (f_A \sqcup (g_B)_i)$.
(6) $0_E \sqsubseteq f_A \sqsubseteq 1_E$.
(7) $(f_A^c)^c = f_A$.
(8) $(\sqcap_{i\in\Delta} (f_A)_i)^c = \sqcup_{i\in\Delta} (f_A)_i^c$.
(9) $(\sqcup_{i\in\Delta} (f_A)_i)^c = \sqcap_{i\in\Delta} (f_A)_i^c$.
(10) If $f_A \sqsubseteq g_B$ then $g_B^c \sqsubseteq f_A^c$.

3. Fuzzy soft topology

Definition 3.1 [3]

A fuzzy soft topology is a map $\tau: \widetilde{(X,E)} \to I$, satisfying the following three axioms:

(1) $\tau(0_E) = \tau(1_E) = 1$,
(2) $\tau(f_A \sqcap g_B) \geq \tau(f_A) \wedge \tau(g_B)$, for all $f_A, g_B \in \widetilde{(X,E)}$,
(3) $\tau(\sqcup_{i\in\Delta} (f_A)_i) \geq \wedge_{i\in\Delta} \tau((f_A)_i)$, for all $((f_A)_i)_{i\in J} \in \widetilde{(X,E)}$.

Then (X, τ, E) is called a fuzzy soft topological space.

Example 3.2 [3]

Let $X = \{x, y\}, E = \{e_1, e_2, e_3\}$, $A = \{e_1, e_2\}$, $B = \{e_2, e_3\}, C = \{e_2\}$. we define f_A, g_B, h_C , $i_E \in \widetilde{(X,E)}$ by

$$f_A = \left\{ f_A(e_1)_{[x,y]}^{[0.6\ ,0.5\ ]} , f_A(e_2)_{[x,y]}^{[0.5\ ,0.4\ ]} \right\},$$
$$g_B = \left\{ g_B(e_2)_{[x,y]}^{[0.5\ ,0.4\ ]} , g_B(e_3)_{[x,y]}^{[0.7\ ,0.2\ ]} \right\},$$
$$h_C = \left\{ h_C(e_2)_{[x,y]}^{[0.5\ ,0.4\ ]} \right\}, \text{ and}$$
$$i_E = \left\{ i_E(e_1)_{[x,y]}^{[0.5\ ,0.4\ ]} , i_E(e_2)_{[x,y]}^{[0.5\ ,0.4\ ]} , i_E(e_3)_{[x,y]}^{[0.7\ ,0.2\ ]} \right\}.$$

We define $\tau: \widetilde{(X,E)} \to I$ by

$$\tau(U)=\begin{cases}1, & U=0_E \ or\ 1_E\ ,\\ 0.6\ , & U=f_A,\\ 0.5\ , & U=g_B,\\ 0.7\ , & U=h_C,\\ 0.5\ , & U=\ i_E,\\ 0\ , & otherwise.\end{cases}$$

Then, (X,τ,E) is a fuzzy soft topological space.

Definition 3.3 [3]

Let (X,τ,E) be a fuzzy soft topological space and $f_A \in \widetilde{(X,E)}$. Then f_A is called fuzzy soft $r-$open (fuzzy soft $r-$closed) if $\tau(f_A) \geq r$ $(\tau(f_A^c) \geq r\ )$,where $r \in I_0$.

Definition 3.4 [3]

Let (X,τ,E) be a fuzzy soft topological space and $f_A \in \widetilde{(X,E)}$, $r \in I_0$. Then fuzzy soft $r-$closure of f_A and fuzzy soft $r-$interior of f_A is denoted by $Cl\ (f_A,r)$ and $Int\ (f_A,r)$, defined by

(1) $Cl\ (f_A,r) = \sqcap \{f_C: \tau(f_C^c) \geq r,\ f_A \sqsubseteq f_C\}$.
(2) $Int\ (f_A,r) = \sqcup \{f_B: \tau(f_B) \geq r,\ f_B \sqsubseteq f_A\}$.

Clearly, $Cl\ (f_A,r)$ is the smallest fuzzy soft $r-$closed which contains f_A and $Int\ (f_A,r)$ is the largest fuzzy soft $r-$open which contained f_A.

Theorem 3.5 [3]

Let (X,τ,E) be a fuzzy soft topological space. For each $f_A, g_B \in \widetilde{(X,E)}$ and $r \in I_0$. Then,

(1) $Cl\ (0_E,r) = 0_E$.
(2) $f_A \sqsubseteq Cl\ (f_A,r)$.
(3) $Cl\ (Cl\ (f_A,r),r) = Cl\ (f_A,r)$.
(4) If $f_A \sqsubseteq g_B$ then $Cl\ (f_A,r) \sqsubseteq Cl\ (g_B,r)$.
(5) f_A is fuzzy soft $r-$closed if and only if $f_A = Cl\ (f_A,r)$.
(6) $Cl\ (f_A \sqcup g_B,r) = Cl\ (f_A,r) \sqcup Cl\ (g_B,r)$.
(7) $Cl\ (f_A \sqcap g_B,r) \sqsubseteq Cl\ (f_A,r) \sqcap Cl\ (g_B,r)$.

Theorem 3.6 [3]

Let (X, τ, E) be a fuzzy soft topological space. For each $f_A, g_B \in \widetilde{(X, E)}$ and $r \in I_0$. Then,

(1) $Int\,(1_E, r) = 1_E$.
(2) $Int\,(f_A, r) \sqsubseteq f_A$.
(3) $Int\,(Int\,(f_A, r), r) = Int\,(f_A, r)$.
(4) If $f_A \sqsubseteq g_B$ then $Int\,(f_A, r) \sqsubseteq Int\,(g_B, r)$.
(5) f_A is fuzzy soft $r-$open if and only if $f_A = Int\,(f_A, r)$.
(6) $Int\,(f_A \sqcap g_B, r) = Int\,(f_A, r) \sqcap Int\,(g_B, r)$.
(7) $Int\,(f_A, r) \sqcup Int\,(g_B, r) \sqsubseteq Int\,(f_A \sqcup g_B, r)$.

4. Fuzzy soft ideal and r-fuzzy soft open local function

In this section, we will introduce the following definitions that we need it later.

Definition 4.1 :

Let (X, τ, E) be a fuzzy soft topological space, for each $f_A \in \widetilde{(X, E)}$, $e_x^\alpha \in FSP\,(X)$ and $r \in I_0$. Then f_A is called $r-$open neighborhood of e_x^α (for short $Q_\tau(e_x^\alpha, r)$) if $e_x^\alpha q f_A$ with $\tau(f_A) \geq r$.

Definition 4.2 :

A mapping $L: \widetilde{(X, E)} \to I$, is called a fuzzy soft ideal on X if it satisfy the following conditions:

(1) $L(0_E) = 1\,, L(1_E) = 0$,
(2) If $f_A \sqsubseteq g_B$, then $L\,(g_B) \leq L\,(f_A)$, for every $f_A, g_B \in \widetilde{(X, E)}$.
(3) $L\,(f_A \sqcup g_B) \geq L\,(f_A) \wedge L\,(g_B)$, for $f_A, g_B \in \widetilde{(X, E)}$.

Then (X, τ, E, L) is called fuzzy soft ideal topological space.

If L_1 and L_2 are fuzzy soft ideals on X, we say that L_1 is finer than L_2 (L_2 is coarser than L_1), denoted by $L_2 \leq L_1$ iff $L_1(f_A) \leq L_1(f_A)$ forevery $f_A \in \widetilde{(X, E)}$.

Definition 4.3 :

Let (X, τ, E, L) be a fuzzy soft ideal topological space and $f_A \in \widetilde{(X, E)}$, then the $r-$fuzzy soft open local function $f^*_{A_r}(\tau, L)$ of f_A is the union of all fuzzy soft points e_x^α such that if $g_B \in Q_\tau(e_x^\alpha, r)$ and $L(h_C) \geq r$, then there is at least one $y \in X$ for which $g_B(e)(y) + f_A(e)(y) - 1 > h_C(e)(y)$.

Let $f_A \in \widetilde{(X,E)}$, we say that f_A is $r-$fuzzy soft open locally in L at e_x^α if there exist $g_B \in Q_\tau(e_x^\alpha, r)$ such that for every $y \in X$, $g_B(e)(y) + f_A(e)(y) - 1 \leq h_C(e)(y)$, for some $L(h_C) \geq r$.
Then, $f_{A_r}^*(\tau, L)$ is the set of all fuzzy soft points at which f_A does not have the property $r-$fuzzy soft open locally.
We will write $f_{A_r}^*$ or $f_{A_r}^*(L)$ instead $f_{A_r}^*(\tau, L)$.

Example 4.4 :

Let (X, τ, E, L) be a fuzzy soft ideal topological space. The simplest fuzzy soft ideal on X is $L^o: \widetilde{(X, E)} \to I$ where

$$L^o(h_C) = \begin{cases} 1, & if\ h_C = 0_E \\ 0, & e.w. \end{cases}$$

If we take $L = L^o$, for each $f_A \in \widetilde{(X, E)}$, we have $f^*_{A_r} = Cl(f_A, r)$.

Theorem 4.5:

Let (X, τ, E) be a fuzzy soft topological space, L_1 and L_2 be two fuzzy soft ideals of X. Then for every $f_A, g_B \in \widetilde{(X, E)}$, $r \in I_0$, we have

(1) If $f_A \le g_B$, then $f^*_{A_r} \le g^*_{B_r}$.
(2) If $L_1 \le L_2$, then $f^*_{A_r}(\tau, L_1) \ge f^*_{A_r}(\tau, L_2)$.
(3) $\left(f^*_{A_r} \vee g^*_{B_r}\right) = (f_A \vee g_B)^*_r$.
(4) If $L(g_B) \ge r$, then $(f_A \vee g_B)^*_r = f^*_{A_r} \vee g^*_{B_r} = f^*_{A_r}$.
(5) $\left(f^*_{A_r} \wedge g^*_{B_r}\right) \ge (f_A \wedge g_B)^*_r$.

Proof :

(1) Suppose that $f_A, g_B \in \widetilde{(X, E)}$ and $r \in I_0$ such that $f^*_{A_r} \nleq g^*_{B_r}$

$\Longrightarrow$ there exists $x \in X$ and $t \in I_0$ such that $f^*_{A_r}(e)(x) \ge t > g^*_{B_r}(e)(x) \dots *$

Since $g^*_{B_r}(e)(x) < t$, there exists $h_D \in Q_\tau(e^\alpha_x, r)$ with $L(i_C) \ge r$ such that for every $y \in X$, we have $h_D(e)(y) + g_B(e)(y) - 1 \le i_C(e)(y)$.

Since $f_A \le g_B$, $h_D(e)(y) + f_A(e)(y) - 1 \le i_C(e)(y)$.

So, $f^*_{A_r}(e)(x) < t$ this is contradiction with $*$.

Hence, $f^*_{A_r} \le g^*_{B_r}$.

(2) Suppose that $f^*_{A_r}(\tau, L_1) \ngeq f^*_{A_r}(\tau, L_2)$, then there exists $x \in X$ and $t \in I_0$, such that $f^*_{A_r}(\tau, L_1)(e)(x) < t \le f^*_{A_r}(\tau, L_2)(e)(x) \dots *$

Since $f^*_{A_r}(\tau, I_1)(e)(x) < t$, then there exists $h_D \in Q_\tau(e^\alpha_x, r)$ with $L_1(i_C) \ge r$ such that for every $y \in X$,

we have $h_D(e)(y) + f_A(e)(y) - 1 \le i_C(e)(y)$.

Since $r \le L_1(i_C) \le L_2(i_C)$, $h_D(e)(y) + f_A(e)(y) - 1 \le i_C(e)(y)$.

Thus, $f^*_{A_r}(I_2, \tau)(e)(x) < t$ it is contradiction with $*$.

Therefore, $f^*_{A_r}(I_1, \tau) \ge f^*_{A_r}(I_2, \tau)$.

(3) $(\Longrightarrow)$ Since $f_A, g_B \le f_A \vee g_B$. By (1) we have

$f^*_{A_r} \le (f_A \vee g_B)^*_r$ and $g^*_{B_r} \le (f_A \vee g_B)^*_r$.

Hence, $f^*_{A_r} \vee g^*_{B_r} \le (f_A \vee g_B)^*_r \dots\dots (1)$

($\Leftarrow$) Suppose that $(f_A \vee g_B)_r^* \nleq f_{A_r}^* \vee g_{B_r}^*$.

Then, there exist $x \in X$ and $t \in I_0$ such that

$\left(f_{A_r}^* \vee g_{B_r}^*\right)(e)(x) < t \leq (f_A \vee g_B)_r^*(e)(x)$ ……$*$

Since $\left(f_{A_r}^* \vee g_{B_r}^*\right)(e)(x) < t \Longrightarrow f_{A_r}^*(e)(x) < t$ and $g_{B_r}^*(e)(x) < t$.

Then, there exists $h_{D_1} \in Q_\tau(e_x^\alpha, r)$such that for every $y \in X$, and for some $L\left(i_{c_1}\right) \geq r$ we have $h_{D_1}(e)(y) + f_A(e)(y) - 1 \leq i_{c_1}(e)(y)$.

Similarly, there exists $j_{D_2} \in Q_\tau(e_x^\alpha, r)$such that for every $y \in X$, and for some $L\left(l_{c_2}\right) \geq r$ we have $j_{D_2}(e)(y) + g_B(e)(y) - 1 \leq l_{c_2}(e)(y)$.

Since $h_{D_1} \wedge j_{D_2} \in Q_\tau(e_x^\alpha, r)$and by Definition 4.3 $L\left(i_{c_1} \vee l_{c_2}\right) \geq r$.

Thus, for every $y \in X$, $\left(h_{D_1} \wedge j_{D_2}\right)(e)(y) + (f_A \vee g_B)(e)(y) - 1 \leq \left(i_{c_1} \vee l_{c_2}\right)(e)(y)$.

Therefore, $(f_A \vee g_B)_r^*(e)(x) < t$, this is contradiction with $*$.

Hence, $f_{A_r}^* \vee g_{B_r}^* \geq (f_A \vee g_B)_r^*$ ….. (2)

We have from (1) and (2) that $f_{A_r}^* \vee g_{B_r}^* = (f_A \vee g_B)_r^*$.

(4) Suppose that $e_x^\alpha \in (f_A \vee g_B)_r^*$, from (3), we get

$e_x^\alpha \in f_{A_r}^* \vee g_{B_r}^* \Longrightarrow e_x^\alpha \in f_{A_r}^*$ *or* $e_x^\alpha \in g_{B_r}^*$

If $e_x^\alpha \in f_{A_r}^* \Longrightarrow (f_A \vee g_B)_r^* \leq f_{A_r}^*$……..(1)

Since $f_A \leq f_A \vee g_B \Longrightarrow f_{A_r}^* \leq (f_A \vee g_B)_r^*$

If $e_x^\alpha \in g_{B_r}^*$, similarly get $(f_A \vee g_B)_r^* \leq g_{B_r}^*$……(2)

From (1) and (2) we have $(f_A \vee g_B)_r^* = f_{A_r}^* \vee g_{B_r}^* = f_{A_r}^*$

(5) Since $f_A \wedge g_B \leq f_A$, then from (1) $(f_A \wedge g_B)_r^* \leq f_{A_r}^*$ ... $(*)$.

Also, $f_A \wedge g_B \leq g_B$, then $(f_A \wedge g_B)_r^* \leq g_{B_r}^*$ ... $(**)$.

From $(*)$ and $(**)$, we get $(f_A \wedge g_B)_r^* \leq f_{A_r}^* \wedge g_{B_r}^*$.

Theorem 4.6 :

Let (X, τ, E, L) be a fuzzy soft ideal topological space and $\left\{f_{A_i}, i \in J\right\} \subset \widetilde{(X, E)}$. Then :

(1) $\left(\vee \left(f_{A_i}\right)_r^* : i \in J\right) \leq \left(\vee f_{A_i} : i \in J\right)_r^*$.

(2) $\left(\wedge f_{A_i} : i \in J\right)_r^* \leq \left(\wedge \left(f_{A_i}\right)_r^* : i \in J\right)$.

Proof :

(1) Since $f_{A_i} \leq \vee f_{A_i}$, $\forall i \in J$, by Theorem 4.5

we get $\left(f_{A_i}\right)_r^* \leq \left(\vee f_{A_i}\right)_r^*$, $\forall i \in J$, then $\vee \left(f_{A_i}\right)_r^* \leq \left(\vee f_{A_i}\right)_r^*$.

(2) Since $\wedge f_{A_i} \leq f_{A_i}, \forall i \in J$, then $\left(\wedge f_{A_i}\right)_r^* \leq \left(f_{A_i}\right)_r^*, \forall i \in J$.

Thus, $\left(\wedge f_{A_i} : i \in J\right)_r^* \leq \left(\wedge \left(f_{A_i}\right)_r^* : i \in J\right)$.

Definition 4.7:

Let (X, τ, E, L) be a fuzzy soft ideal topological space and $f_A \in \widetilde{(X, E)}$. We define the fuzzy soft closure operator $Cl^*(f_A, r)$ and fuzzy soft interior operator denoted by $Int^*(f_A, r)$, as follows

$$Cl^*(f_A, r) = f_A \vee f_{A_r}^* \ , \ Int^*(f_A, r) = f_A \wedge ((f_A^c)_r^*)^c.$$

Also, $\tau^*(I)$ is called a fuzzy soft topological, generated by $Cl^*(f_A, r)$ such that $\tau^*(I)(f_A) = \vee \{r : Cl^*(f_A^c, r) = f_A^c\}$.

If $L = L^0$, then $Cl^*(f_A, r) = f_A \vee f_{A_r}^* = f_A \vee Cl(f_A, r) = Cl(f_A, r)$

For $f_A \in \widetilde{(X, E)}$. So $\tau^*(L^0) = \tau$.

Theorem 4.8 :

Let (X, τ, E, L) be a fuzzy soft ideal topological space and $f_A, g_B \in \widetilde{(X, E)}$ and $r \in I_0$. Then :

(1) $Cl^*(f_A^c, r) = \left(Int^*(f_A, r)\right)^c$ and $\left(Cl^*(f_A, r)\right)^c = Int^*(f_A^c, r)$.

(2) $Int^*(f_A \wedge g_B, r) = Int^*(f_A, r) \wedge Int^*(g_B, r)$.

(3) $Int^*(f_A \vee g_B, r) \leq Int^*(f_A, r) \vee Int^*(g_B, r)$.

(4) $Int(f_A, r) \leq Int^*(f_A, r) \leq f_A \leq Cl^*(f_A, r) \leq Cl(f_A, r)$.

Proof :

(1) $Cl^*(f_A^c, r) = f_A^c \vee (f_A^c)_r^* = f_A^c \vee (((f_A^c)_r^*)^c)^c = (f_A \wedge ((f_A^c)_r^*)^c)^c$

$= \left(Int^*(f_A, r)\right)^c$.

(2) $Int^*(f_A \wedge g_B, r) = (f_A \wedge g_B) \wedge (((f_A \wedge g_B)^c)_r^*)^c$

$= (f_A \wedge g_B) \wedge ((f_A^c \vee g_B^c)_r^*)^c = (f_A \wedge g_B) \wedge \left(((f_A^c)_r^* \vee (g_B^c)_r^*)\right)^c$

$= (f_A \wedge g_B) \wedge ((f_A^c)_r^*)^c \wedge ((g_B^c)_r^*)^c$

$= [f_A \wedge ((f_A^c)_r^*)^c] \wedge [g_B \wedge ((g_B^c)_r^*)^c] = Int^*(f_A, r) \wedge Int^*(g_B, r)$.

(3) and (4) follows from definitions of Int^*, Cl^* and Cl.

Theorem 4.9 :

Let (X, τ_1, E, I) and (X, τ_2, E, I) are fuzzy soft ideal topological spaces and $\tau_1 \leq \tau_2$. Then:

(1) $f_{A_r}^*(\tau_2, L) \leq f_{A_r}^*(\tau_1, L)$,

(2) $\tau_1^*(L) \leq \tau_2^*(L)$.

Proof :

(1) Suppose that $f_{A_r}^*(\tau_2, L) \nleq f_{A_r}^*(\tau_1, L)$, then there exist $x \in X$ and $t \in I_0$ such that $f_{A_r}^*(\tau_2, L)(e)(x) \geq t > f_{A_r}^*(\tau_1, L)(e)(x)$ ……*

Since $f_{A_r}^*(\tau_1, L)(e)(x) < t$, there exists $h_C \in Q_{\tau_1}(e_x^\alpha, r)$ with $L(j_D) \geq r$ such that for every $y \in X$, we have $h_C(e)(y) + f_A(e)(y) - 1 \leq j_D(e)(y)$.

Since $\tau_1 \leq \tau_2, h_C \in Q_{\tau_2}(e_x^\alpha, r)$.

Thus, $f_{A_r}^*(\tau_2, L)(e)(x) < t$, and this contradiction with $*$.

Hence, $f_{A_r}^*(\tau_2, L) \leq f_{A_r}^*(\tau_1, L)$.

(2) Suppose that $f_A \in \tau_1^*(L)$ by definition of τ^*,

we have $Cl_{\tau_1}^*(f_A^c, r) = f_A^c$ then $f_A^c \vee (f_A^c)_r^* = f_A^c$ and

$Cl_{\tau_2}^*(f_A^c, r) = f_A^c \vee (f_A^c)_r^*$.

From (1), $f_A^c \vee (f_A^c)_r^*(\tau_2) \leq f_A^c \vee (f_A^c)_r^*(\tau_1) = f_A^c$, then

$f_A^c \vee (f_A^c)_r^*(\tau_2) \leq f_A^c \Longrightarrow Cl_{\tau_2}^*(f_A^c, r) \leq f_A^c$.

Since $f_A^c \leq Cl_{\tau_2}^*(f_A^c, r)$.

Hence, $f_A^c = Cl_{\tau_2}^*(f_A^c, r)$.

Therefore, $f_A \in \tau_2^*(L)$.

So, $\tau_1^*(L) \leq \tau_2^*(L)$.

Theorem 4.10 :

Let (X, τ, E, L_1) and (X, τ, E, L_2) are two fuzzy soft ideal topological spaces and $L_1 \leq L_2$. Then:

(1) $f_{A_r}^*(\tau, L_1) \geq f_{A_r}^*(\tau, L_2)$,

(2) $\tau^*(L_1) \leq \tau^*(L_2)$.

Proof :

1) Suppose that $f_{A_r}^*(\tau, L_1) \ngeq f_{A_r}^*(\tau, L_2)$, then there exist $x \in X$ and $t \in I_0$ such that $f_{A_r}^*(\tau, L_1)(e)(x) < t \leq f_{A_r}^*(\tau, L_2)(e)(x)$ … *

Since $f_{A_r}^*(\tau, L_1)(e)(x) < t$, there exists $h_D \in Q_\tau(e_x^\alpha, r)$with $L_1(j_C) \geq r$ such that for every $y \in X$, we have $h_D(e)(y) + f_A(e)(y) - 1 \leq j_C(e)(y)$.

Since $L_1 \leq L_2$, then $L_2(j_C) \geq r$

Thus, $f_{A_r}^*(\tau, L_2)(e)(x) < t$ and this contradiction with $*$.

Therefore, $f_{A_r}^*(\tau, L_1) \geq f_{A_r}^*(\tau, L_2)$.

2) Let $f_A \in \tau^*(L_1)$ by definition 4.7 $\Longrightarrow Cl^*({f_A}^c, r) = {f_A}^c$

$\Longrightarrow {f_A}^c \vee ({f_A}^c)_r^*(L_1) = {f_A}^c$.

Now, $Cl^*({f_A}^c, r) = {f_A}^c \vee ({f_A}^c)_r^*(L_2)$

Since $L_1 \leq L_2$, by Theorem 4.5 (2), we get

$({f_A}^c)^*_r(L_2) \le ({f_A}^c)^*_r(L_1) \implies ({f_A}^c)^*_r(L_2) \le {f_A}^c$

$\implies {f_A}^c \vee ({f_A}^c)^*_r(L_2) = {f_A}^c$

$\implies f_A \in \tau^*(L_2)$ and therefore, $\tau^*(L_1) \le \tau^*(L_2)$.

Definition 4.11 :

Let θ be a subset of $\widetilde{(X,E)}$ and $0_E \notin \theta$. Then a mapping $\beta: \theta \to I$ is called a fuzzy soft base on X if the following condition satisfy:

(1) $\beta(1_E) = 1$,

(2) $\beta(f_A \wedge g_B) \ge \beta(f_A) \wedge \beta(g_B)$, for all $f_A, g_B \in \theta$.

Theorem 4.12:

Let θ be a subset of $\widetilde{(X,E)}$ and $0_E \notin \theta$. We define a mapping $\beta: \theta \to I$ on X as following $\beta(f_A) = \vee\{\tau(g_B) \wedge I(h_D) \ : \ f_A = g_B \wedge h_D^c\}$.

Then β is base for the fuzzy soft topology τ^*.

Proof :

1) Since $L(0_E) = 1$ then $\beta(1_E) = 1$.

2) Suppose that $f_A, g_B \in \theta$ such that $\beta(f_A \wedge g_B) \ngeq \beta(f_A) \wedge \beta(g_B)$.

There exists $t \in I_0$, such that $\beta(f_A \wedge g_B) < t \le \beta(f_A) \wedge \beta(g_B)$ ... $*$

$t \le \beta(f_A) \wedge \beta(g_B)$ then $t \le \beta(f_A)$ and $t \le \beta(g_B)$, there exist $i_C, h_D, j_M, l_N \in \theta$, whit $f_A = i_C \wedge j_M^c$ and $g_B = h_D \wedge l_N^c$.

Such that $\beta(f_A) \ge \tau(i_C) \wedge L(j_M) \ge t$ and

$\beta(g_B) \ge \tau(h_D) \wedge L(l_N) \ge t$

$$f_A \wedge g_B = (i_C \wedge j_M^c) \wedge (h_D \wedge l_N^c) = (i_C \wedge h_D) \wedge (j_M^c \wedge l_N^c)$$
$$= (i_C \wedge h_D) \wedge (j_M \vee l_N)^c.$$

Hence, $\beta(f_A \wedge g_B) \ge \tau(i_C \wedge h_D) \wedge L(j_M \vee l_N)$

$$\ge \tau(i_C) \wedge \tau(h_D) \wedge L(j_M) \wedge L(l_N)$$
$$= \big(\tau(i_C) \wedge L(j_M)\big) \wedge \big(\tau(h_D) \wedge L(l_N)\big) \ge t$$

$\beta(f_A \wedge g_B) \ge t$ this is contradiction with $*$.

Therefore, $\beta(f_A \wedge g_B) \ge \beta(f_A) \wedge \beta(g_B)$.

Then β is a base of a fuzzy soft topology τ^*.

Theorem 4.13 :

Let (X, τ, E) be a fuzzy soft topological space and L_1, L_2 are two fuzzy soft ideals on X, then for any $f_A \in \widetilde{(X,E)}$ and $r \in I_0$,

(1) $f^*_{A_r}(\tau, L_1 \wedge L_2) = f^*_{A_r}(\tau, L_1) \vee f^*_{A_r}(\tau, L_2)$,

(2) $f^*_{A_r}(\tau, L_1 \vee L_2) = f^*_{A_r}(\tau^*(I_2), L_1,) \wedge f^*_{A_r}(\tau^*(I_1), L_2)$.

Proof :

(1) Suppose that $f^*_{A_r}(\tau, L_1 \wedge L_2) \nleq f^*_{A_r}(\tau, L_1) \vee f^*_{A_r}(\tau, L_2)$, then there exist $x \in X, t \in I_0$ such that

$f^*_{A_r}(\tau, L_1 \wedge L_2)(e)(x) \geq t > f^*_{A_r}(\tau, L_1)(e)(x) \vee f^*_{A_r}(\tau, L_2)(e)(x) \ldots *$

Since $f^*_{A_r}(\tau, L_1)(e)(x) \vee f^*_{A_r}(\tau, L_2)(e)(x) < t$

$f^*_{A_r}(\tau, L_1)(e)(x) < t$ and $f^*_{A_r}(\tau, L_2)(e)(x) < t$

Since $f^*_{A_r}(\tau, L_1)(e)(x) < t$, there exists $h_D \in Q_\tau(e^\alpha_x, r)$ and for some $L_1(i_C) \geq r$ such that for every $y \in X$, we have $h_D(e)(y) + f_A(e)(y) - 1 \leq i_C(e)(y)$.

Since $f^*_{A_r}(\tau, L_2)(e)(x) < t$, there exists $j_M \in Q_\tau(e^\alpha_x, r)$ and for some $L_2(l_N) \geq r$ such that for every $y \in X$, we have $j_M(e)(y) + f_A(e)(y) - 1 \leq l_N(e)(y)$.

Therefore, $(h_D \wedge j_M)(e)(y) + f_A(e)(y) - 1 \leq (i_C \wedge l_N)(e)(y)$, for every $y \in X$.

Since $h_D \wedge j_M \in Q_\tau(e^\alpha_x, r)$ and $L_1 \wedge L_2\ (i_C \wedge l_N) \geq r$.

Hence, $f^*_{A_r}(\tau, L_1 \wedge L_2) < t$, this contradiction with $*$.

So, $f^*_{A_r}(\tau, L_1 \wedge L_2) \leq f^*_{A_r}(\tau, L_1) \vee f^*_{A_r}(\tau, L_2)$.

Also, since $L_1, L_2 \geq L_1 \wedge L_2$ by Theorem 4.5 (2) then

$f^*_{A_r}(L_1 \wedge L_2) \geq f^*_{A_r}(L_1)$ and $f^*_{A_r}(\tau, L_1 \wedge L_2) \geq f^*_{A_r}(L_2)$

$f^*_{A_r}(L_1 \wedge L_2) \geq f^*_{A_r}(L_1) \vee f^*_{A_r}(L_2)$.

Then, $f^*_{A_r}(\tau, L_1 \wedge L_2) = f^*_{A_r}(\tau, L_1) \vee f^*_{A_r}(\tau, L_2)$.

(2) Suppose that $f^*_{A_r}(\tau, L_1 \vee L_2) \ngeq f^*_{A_r}(\tau^*(L_2), L_1) \wedge f^*_{A_r}(\tau^*(L_1), L_2)$, then there exist $x \in X, t \in I_0$ such that

$$f^*_{A_r}(\tau, L_1 \vee L_2)(e)(x) < t \leq f^*_{A_r}(\tau^*(L_2), L_1)(e)(x) \wedge f^*_{A_r}(\tau^*(L_1), L_2)(e)(x) \ldots *$$

Since $f^*_{A_r}(\tau, L_1 \vee L_2)(e)(x) < t$, there exists $h_D \in Q_\tau(e^\alpha_x, r)$ and for some $(L_1 \vee L_2)(g_B) \geq r$ such that for every $y \in X$, we have $h_D(e)(y) + f_A(e)(y) - 1 \leq g_B(e)(y)$.

Since $\tau \leq \tau^*$ we can find $j_M \in Q_{\tau^*(L_1)}(e^\alpha_x, r)$ or $i_N \in Q_{\tau^*(L_2)}(e^\alpha_x, r)$ such that for every $y \in X$, and for some $L_1(l_R) \geq r$ or $L_2(p_T) \geq r$ we have $j_M(e)(y) + f_A(e)(y) - 1 \leq l_R(e)(y)$ or

$i_N(e)(y) + f_A(e)(y) - 1 \leq p_T(e)(y)$.

This implies that $f^*_{A_r}\big(L_2, \tau^*(I_1)\big)(e)(x) < t$ or $f^*_{A_r}\big(L_1, \tau^*(I_2)\big)(e)(x) < t$ this contradiction with $*$.

Thus, $f^*_{A_r}(\tau , L_1 \vee L_2) \geq f^*_{A_r}(\tau^*(L_2) , L_1) \wedge f^*_{A_r}(\tau^*(L_1) , L_2)$.

Similarly, we can prove

$f^*_{A_r}(\tau, L_1 \vee L_2) \leq f^*_{A_r}(\tau^*(L_2) , L_1) \wedge f^*_{A_r}(\tau^*(L_1) , L_2)$.

Theorem 4.14 :

Let (X, τ, E, L) be a fuzzy soft ideal topological space. For any $f_A \in \widetilde{(X, E)}$ and $r \in I_0$,then $f^*_{A_r}(L) = f^*_{A_r}(\tau^* , L)$ and $\tau^*(L) = \tau^{**}$.

Proof:

By putting $L_1 = L_2$ and using Theorem 4.13, we have required result.

References:

[1] Cetkin V., and Aygun H., A note of fuzzy topological spaces, Proceedings of the 8th Conference of European Society for Fuzzy Logic and Technology (EUSFLAT 2013), 56-60.

[2] Chang C. L., Fuzzy topological spaces, J. Math. Anal.Appl, 24 (1968), 182-190.

[3] Kalpana G. and Kalaivani C., Fuzzy soft topology, International Journal of Engineering Studies, Vol. 9, No.1 (2017), 45-56.

[4] Maji P.K., Biswas R. and Roy A. R., Fuzzy soft sets, J. Fuzzy Math. 9(3) (2011) 589-602.

[5] Ming P. P. and Ming L. Y., Fuzzy topology I.Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571-599.

[6] Molodtsov D., "Soft Set Theory-First results", Computers Math. Appl., 37(4/5)(1999), 19-31.

[7] Roy S. and Samanta T.K., A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform., 3(2) (2011)305-311.

[8] Saber Y. M. and Abdel-Sattar M. A. "Ideal On Fuzzy Topological spaces" Applied Mathematical Sciences ,Vol. 8, (2014), no. 34, 1667-1691.

[9] Serkan A. and Idris Z., On fuzzy soft topological spaces, Annals of Fuzzy Mathematics and Informatics, Vol. 5, No. 2, (March 2013), 377-386.

[10] Shabir M. and Naz M., On soft topological spaces, Computers and Mathematics with Applications, 61(7)(2011), 1786-1799.

[11] Sostak A.P. ,On fuzzy topological structure , Suppl. Rend. Circ. Matem. Palerms Ser.II, 11(1985)89-103.

[12] Tany B. and Kandemir M. B., Topological structures of fuzzy soft sets, Compu. Math. Appl., 61(2011) 412-418.
[13] Varol B.P. and Aygun H., Fuzzy soft topology, Hacettepe Journal of Mathematics and Statistics, 41(3)(2012), 407-419.
[14] Zadah L.A.,"Fuzzy sets" ,Information and control ,8.(1965).