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Three-Species Lotka-Volterra Food Chain Model with Fear Effect and Hunting Cooperation

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Abstract

From an ecological standpoint, the dynamics of the food chain model under the fear that is thus placed on the predator population and the inuence of cooperative hunting among them are quite significant. In the current paper, we use the food chain Lotka-Volterra predator-prey model to examine the effects of hunting cooperation among middle predators and top-predator-induced fear in the middle predator population. The system has four points of equilibrium. The model's dynamical characteristics are studied. It has been discovered that dynamical behavior is quite sensitive to parameter values and displays a range of dynamics without degenerating into chaos and that the presence of cooperation and fear signicantly affects the stability of the dynamics of the system. Computer simulations have been used to illustrate the theoretical results.

Keywords: food chain model , fear effect , hunting cooperation .

1. Introduction

The most significant biological processes in ecology and population biology include interactions between organisms and their environment, as well as the evolution of species [17]. Different types of models have been constructed and examined since mathematical modeling is a potent tool for examining the aforementioned biological processes [18]. The first population biology model was developed by Malthus [16], and Verhulst [28] later improved it for a more realistic situation. Historically, modeling projects for food chain dynamics have been on for years. A straightforward model of interacting species by Lotka [15] and Volterra [29] that still bears their shared names was independently created and described the beginning

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of the cycle in biological populations. Later, the Holling type-II functional response for the predator and a logistic growth term for the prey were added to the Lotka-Volterra model [10].

It is common knowledge that interactions between predators and prey have been primarily inuenced by predation. In the wild, a predator kills and consumes its prey. Nevertheless, mounting evidence suggests that many animals may also assess the risk of predation, changing their behavior [14]. Despite not involving actual killing, the fear of predators on prey nonetheless has an impact on the dynamics of both the predator and the prey populations. Additionally, there is proof that this indirect inuence may be just as significant as the direct effect [11]. Because of their fear of predators, prey populations may move their grazing areas to safer places, sacrifice the regions with the highest intake rates, enhance alertness, change their sexual and reproductive habits,etc [7, 24]. Several predator-prey interactions have been studied by many researchers, including those between elk and wolves [5], snowshoe hares and dogs [26], and mule deer and mountain lions [13].

Animal social existence depends heavily on cooperation, which is also critical to biological processes. Cooperation in the context of hunting can be as simple as two or more peoplekin or non-kin improving their fitness by working together to achieve a common objective. Group hunting has many benefits, including an increase in hunting success rate with the number of adults [6], a reduction in chasing distance [6], an increase in the likelihood of catching large prey [2], the ability to find food more quickly as group size increases [23], along with their hunting tactics, prey kinds to target, whether they targeted one or more prey at once, and the percentage of success they achieved. Predators may engage in cooperative and coordinated attacks on both solitary and huddled prey. During hunting, a variety of animals exhibit cooperative behavior, such as wild dogs [6], birds [9], lions [27], etc .

Using a mathematical modeling approach, many authors investigated the effects of hunting cooperation and the fear effect in the predator-prey system separately [1, 20, 30] and others. Later, Pal et al. [19] investigated the impact of cooperation and fear effect in a predator-prey model simultaneously. They observed that the hunting cooperation among predators induces fear in prey population and as a result birth rate of prey population reduces. Also, regarding the three-species food chain model, Duarte et al. [8] investigated of a three-species food chain model including the predator's participation in hunting. According to Panday et al. [21] investigated of a three-species food chain model, taking into account how the growth rate of the intermediate predator is lowered as a result of the cost of fear of the top predator and how the growth rate of the prey is reduced as a result ofthe cost of dread of the middle predator. They noticed that fear has the power to stabilize a chaotic system.

Recently, Cong et al. [3] study the dynamics of a three-species food chain model with fear effect. They derived the predators functional response by using the classical Hollings type II. The results show that the predators fear effect can transform the system from chaotic dynamics to a stable state. The purpose of this study : To give an ecological model describes the impact of the effect of fear and hunting cooperation in a three-species food chain model. And to study the dynamic of this model as an application to the continuous dynamical systems. We presented our model with effect of fear and hunting cooperation in section 2. Basic model characteristics like positivity and boundedness are covered in section 3. The model's equilibria are studied in section 4 of the text. In Section 5, the local stability analysis and the hopf bifuraction analysis of the model are examined. In Sec- tion 6,the numerical simulations are carried out. The paper concludes with a conclusion in Section 7.

2. Mathematical Model

In this section, a Lotka-Volterra food chain prey-predator model has been developed. Indeed, the two-dimensional Lotka-Volterra model represents a simple prey-predator food chain model. Also, the following model is the conventional Lotka-Volterra model of a three-dimensional food chain model that has been presented in numerous research papers,

including [12, 22]:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - d_1x - g_1(x,y), \\ \frac{dy}{dt} &= \beta g_1(x,y) - d_2y - g_2(y,z), \\ \frac{dz}{dt} &= \delta g_2(y,z) - d_3z. \end{aligned} \tag{2.1}$$

Where the functions response $g_1(x,y)$ and $g_2(y,z)$ are given by mxy and γyz. The nonnegative variables x, y, z are the population density of prey, middle predator and top predator, respectively. The systems (2.1) parameters, which are specified as being positive, are described as follows:the parameter r is the birth rate of the prey, the parameters d_1, d_2, and d_3 are the natural death rates of prey, middle predator and top predator respectively, the parameter m is the attack rate of middle predator on the prey, the parameter m is the attack rate of the top predator on middle predator, the parameter γ is a conversion rate of prey to middle predator, the parameter β is a conversion rate of predator to top predator, the parameter k is environmental carrying capacities of prey. According to [1], hunting cooperation in predators can rise predator production. Also, as stated by [3], the fear effect can inuence middle predator production. Thus, we incorporate the hunting cooperation and fear effect for middle predator, by modify $g_1(x,y)$with $\frac{(m+py)xy}{(1+ez)}$, where, p is the parameter describing middle predator cooperation during hunting and e is the level of fear due to top predator. Hence, with the above assumptions the dynamic of the described a three-species food chain system affected by the fear effect and hunting cooperation can be represented mathematically by the following set of nonlinear differential equations.

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right) - d_1 x - \frac{(m+py)xy}{1+ez}, \\ \frac{dy}{dt} &= \frac{\beta(m+py)xy}{1+ez} - d_2 y - \gamma yz, \\ \frac{dz}{dt} &= \delta\gamma yz - d_3 z. \end{aligned} \tag{2.2}$$

Here, the model (2.2) has been analysed with the initial conditions $x(0) \geq 0;\ y(0) \geq\ 0\ and\ z(0) \geq 0$. Also, the right-hand side functions are Lipschitz functions since they are continuous and have continuous partial derivatives. As a result, the system (2.2) has a solution that is both existing and unique.

3. Positivity and Boundedness of the Solutions

The positivity and boundedness of the solutions of a biological model is important because it indicates the ecological meaningfulness of the system. Any solutions initiating from an interior point in first octant always remain in it. Therefore, the test of positivity and boundedness of a biological model system is necessary and so we are interested to prove.

Theorem 1: The solutions of the model system (2.1) that satisfy the initial condition $(x(0) > 0, y(0) > 0, z(0), 0)$, which initiate in R_+^3 are positive.

Proof : To prove the positivity of the system, we define $\zeta = \{(x, y, z) \in R_+^3 : x > 0, y > 0, z > 0\}$, according to ecological significance. Integrate the equations of the system (2.1) using the initial conditions $x(0)\ >\ 0;\ y(0)\ >\ 0\ and\ z(0)\ >\ 0$

which gives the following :

$$\begin{aligned} x(t) &= x(0)e^{\int_0^t\left(r\left(1-\frac{x}{k}\right)-d_1-\frac{(m+py)y}{1+ez}\right)ds}, \\ y(t) &= y(0)e^{\int_0^t\left(\frac{\beta(m+py)x}{1+ez}-d_2-\gamma z\right)ds}, \\ z(t) &= z(0)e^{\int_0^t(\delta\gamma yz-d_3 z)ds}. \end{aligned} \tag{3.1}$$

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Since the parts in the right hand side are positive for all positive initial conditions, hence all the solutions of the system which start from the first octant remain also in first quadrant. It completes the proof of positivity of the solutions of the system. ∎

Theorem 2: All the solutions of the system (2.1)that satisfy the initial condition $(x(0) \geq 0, y(0) \geq 0; Z(0) \geq 0)$, which initiate R_+^3 are bounded .

Proof : Let $(x(t);\ y(t);\ z(t))$ is an arbitrary solution of the system (2.1) with an assumed non-negative initial condition.

let us consider a function $W(x(t), y(t), z(t)) = x(t) + y(t) + z(t)$,Then :

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}.$$

Let us choose a positive constant $\rho > 0$ such that , $\rho = \min\{r, d_1, d_2, d_3\}$ then may get

$$\frac{dW}{dt} + \rho W \leq \frac{k(r - d_1 + \rho)^2}{4r}$$

Now assume that $H = \frac{k(r-d_1+\rho)^2}{4r}$. Then,

$$\frac{dW}{dt} + \rho W \leq H$$

The following results are derived by applying Gronwall's Inequality:

$$0 \leq W\big(x(t), y(t), z(t)\big) \leq \frac{H}{\rho}\Big((1 - e^{-\rho t}) + W\big(x(0), y(0), z(0)\big)\Big)\, e^{-\rho t}$$

which implies ,

$$\lim_{t\to\infty} W\big(x(t), y(t), z(t)\big) \leq \frac{H}{\rho}$$

Then,

$$0 \leq W\big(x(t), y(t), z(t)\big) \leq \frac{H}{\rho}, \forall t > 0$$

Thus, the proof is successful and all of system (1) solutions are uniformly bounded. ■

4. Existence of equilibria

A food chain model with fear effect and hunting cooperation given by the system (2.1) has four types of nonnegative equilibrium points, namely $E_i, i = 0,1,2,3$. The equilibrium points can be summed up as follows if we take into account the righthand sides of system (2.1).

1- $E_0 = (0,0,0)$ is the trivial equilibrium point, that is always exists.

2- $E_1 = (x_1, 0,0)$ is the predators-free equilibrium point, where x_1 exists in the interior of R_+ if and only if x_1 represent the positive solution of the equation:

$$\left(r - \frac{rx_1}{k}\right) - d_1 = 0 \tag{4.1}$$

From Eq.(4.1) we get ,

$$x_1 = \frac{k(r-d_1)}{r} \tag{4.2}$$

Now, from Eq.(4.2) and the positivity of x_1, the below condition yields:

$$r > d_1 \tag{4.3}$$

So, (4.2) and (4.3) are represent the necessary conditions for the existence of the equilibrium point E_1.

3- $E_2 = (x_2, y_2, 0)$ is the top predator-free equilibrium point ,where (x_2, y_2) exists in the interior of R^2_+, if and only if, x_2 and y_2 are the positive roots of the following set of algebraic equations:

$$r\left(1 - \frac{x_2}{k}\right) - d_1 - (m + py_2)y_2 = 0 \tag{4.4}$$

$$\beta(m + py_2)x_2 - d_2 = 0 \tag{4.5}$$

From Eq.(4.5) we obtain the value of x_2

$$x_2 = \frac{d_2}{\beta(m + py_2)} \tag{4.6}$$

Further, if we substituting the value of x_2 into Eq.(4.4), can then obtain:

$$c_1 y_2^3 + c_2 y_2^2 + c_3 y_2 + c_4 = 0 \tag{4.7}$$

where,

$$c_1 = p^2 k\beta > 0,$$

$$c_2 = 2mpk\beta > 0,$$

$$c_3 = k\beta\big((d_1 - r)p + m^2\big),$$

$$c_4 = (d_1 - r)km\beta + rd_2.$$

Now, let $k = \frac{d_2}{\beta m}$, then by Descartes rule of signs when condition (4.3) holds, Eq.(4.7) may has positive solutions in two cases:

- Case 1: if $c_4 < 0$, then Eq.(4.7) has a unique positive solution and this takes place when:

$$k < \frac{k(r - d_1)}{r} \tag{4.8}$$

  Here, we have $x_2 < \frac{k(r-d_1)}{r}$

- Case 2: if $c_4 > 0$ and $c_3 < 0$, then Eq.(4.7) may has a two positive solution and this takes place when:

$$k > \frac{k(r - d_1)}{r} \tag{4.9}$$

$$p > \frac{m^2}{(r - d_1)} \tag{4.10}$$

So, the values of k and p in Eq.(4.8), Eq.(4.9) and Eq.(4.10) determine the necessary conditions for the existence of the top predator-free equilibrium point. These circumstances demonstrate that cooperative hunting has opposing effects on the populations of species x and y; as a result, increases in cooperative hunting may result in decreased numbers of species x and increased numbers of species y, and vice versa.

4- $E_3 = (x_3, y_3, z_3)$ is the positive interior equilibrium point, that exists in the interior of R_+^3 , if and only if, x_3, y_3 and z_3 are the positive solutions of the following set of equations:

$$r\left(1 - \frac{x_3}{k}\right) - d_1 - \frac{(m + py_3)y_3}{1 + ez_3} = 0 \tag{4.11}$$

$$\frac{\beta(m+py_3)x_3}{1+ez_3}-d_2-\gamma z_3=0 \tag{4.12}$$

$$\delta\gamma y_3-d_3=0 \tag{4.13}$$

From Eq.(4.12), x_3 may given by:

$$x_3=\frac{(d_2+\gamma z_3)(1+ez_3)}{\beta(m+py_3)} \tag{4.14}$$

and from Eq.(4.13), y_3 can given by:

$$y_3=\frac{y_3}{\delta\gamma} \tag{4.15}$$

Further, by substituting Eq.(4.14) and Eq.(4.15) in Eq.(4.11), may have the following polynomial equation:

$$s_1z_3^3+s_2z_3^2+s_3z_3+s_4=0 \tag{4.16}$$

where,

$$s_1=re^2\gamma>0$$

$$s_2=re(2ed_2+\gamma)>0$$

$$s_3=re(2ed_2+\gamma)-k\beta e(r+d_1)(m+py_3)$$

$$s_4=rd_2+k\beta(m+py_3)[y_3(m+py_3)-(r-d_1)].$$

When condition (4.3) is true, the following two situations of Eq.(4.16) may have positive solutions according to Descartes' rule of signs:

- Case 1: If the inequality $s_4<0$ is true, this confers
$$\frac{d_2}{\beta(m+py_3)}+\frac{k(m+py_3)y_3}{r}<\frac{k(r-d_1)}{r} \tag{4.17}$$
So, condition (4.17) gains that Eq.(4.16) has only one positive solution. Hence system (2.1) has a unique positive equilibrium point.
- Case 2: If $s_3<0$ and $s_4>0$ are hold, and this might occur when
$$\frac{d_2}{\beta(m+py_3)}+\frac{k(m+py_3)y_3}{r}>\frac{k(r-d_1)}{r} \tag{4.18}$$
$$(m+py_3)>\frac{r(2ed_2+\gamma)}{k\beta e(r+d_1)} \tag{4.19}$$

So, conditions (4.18) and (4.19) earn that Eq.(4.16) has two positive solutions. Hence, in this case, system (2.1) has two positive equilibrium points. In light of the aforementioned two situations, it can be shown that p has a complex impact on the existence of E_3.

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5. Local stability and Hopf Bifurcation

In this section, the local dynamic behaviour of the system (2.1) around each of the above equilibrium points is discussed by making use of the eigenvalue method. Also, Hopf bifurcation at positive interior equilibrium point E_3 is demonstrated. At any point (x, y, z), system (2.1) has the following Jacobian matrix:

$$J(x,y,z) = \begin{bmatrix} r - \frac{2rx}{k} - d_1 - \frac{(m+py)y}{1+ez} & -\frac{(m+2py)x}{1+ez} & \frac{e(m+py)xy}{(1+ez)^2} \\ \frac{\beta(m+py)y}{1+ez} & \frac{\beta(m+2py)x}{1+ez} & -\left(\frac{\beta e(m+py)xy}{(1+ez)^2} + \gamma y\right) \\ 0 & \delta\gamma z & \delta\gamma y - d_3 \end{bmatrix} \quad (5.1)$$

and the stability of the equilibrium points is assessed using the eigenvalues of this matrix at E_0; E_1; E_2; and E_3 , which are obtained as the roots of the characteristic equations.

<u>**Theorem 5.1**</u>: If $r < d_1$, the trivial equilibrium point $E_0 = (0,0,0)$ is asymptotically stable; otherwise E_0 a saddle point if it is hyperbolic.

Proof : The Jacobian matrix (5.1) at E_0 can be written by:

$$J(E_0) = \begin{bmatrix} r - d_1 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix} \quad (5.2)$$

The diagonal matrix J(E_0) has:

$$\lambda_{01} = r - d_1,$$

$$\lambda_{02} = -d_2,$$

$$\lambda_{03} = -d_3,$$

where, $\lambda_{0i}, i1,2,3,$ represent the eigenvalues that describe the dynamics in the directions of their eigenvectors. Hence, the equilibrium point E_0 is asymptotically stable if $r < d_1$, and saddle point if the prey survival condition $r > d_1$ persist.

■

<u>**Theorem 5.2**</u>: The equilibrium point $E_1 = (x_1, 0,0)$ is local asymptotically stable, when-ever:

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$$x_1 < \frac{d_2}{\beta m}. \tag{5.3}$$

Proof : At the equilibrium point E_1, the Jacobian matrix (5.1) might be abbreviated to:

$$J(E_1) = \begin{bmatrix} d_1 - r & -\frac{mk(r-d_1)}{r} & 0 \\ 0 & \frac{\beta mk(r-d_1)}{r} - d_2 & 0 \\ 0 & 0 & -d_3 \end{bmatrix} \tag{5.4}$$

Matrix $J(E_1)$ has the following eigenvalues:

$$\lambda_{11} = d_1 - r < 0,$$

$$\lambda_{12} = \frac{\beta mk(r-d_1)}{r} - d_2,$$

$$\lambda_{13} = -d_3 < 0.$$

Obviously, if the following is true, λ_{12} is negative

$$\frac{k(r-d_1)}{r} < \frac{d_2}{\beta m}. \tag{5.5}$$

Then, $J(E_1)$ has only negative eigenvalues. Thus, we can state that, E_1 is locally asymptotical stable if (5.3) is still met. But, when (5.3) is not valid, E_1 becomes a saddle point owing to $\lambda_{12} > 0$. ■

Theorem 5.3: The equilibrium point $E_2 = (x_2, y_2, 0)$ is locally asymptotical stable, when- ever the next two conditions are hold:

$$x_2 < \frac{k(m+py_2)(m+2py_2)}{rp}, \tag{5.6}$$

$$y_2 < \min\left\{\frac{d_3}{\delta\gamma}, \frac{r}{\beta kp}\right\}. \tag{5.7}$$

Proof : At E_2, the Jacobian matrix of system (2.1) may be reduced to:

$$J(E_2) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \tag{5.8}$$

where,

$$a_{11} = -\frac{rx_2}{k} < 0 \quad , \quad a_{21} = \beta(m + py_2)y_2 > 0,$$

$$a_{12} = -x_2(m + 2py_2) < 0 \quad , \quad a_{22} = \beta p x_2 y_2 > 0,$$

$$a_{13} = \frac{ed_2}{\beta} y_2 > 0 \quad , \quad a_{23} = -(ed_2 + \gamma)y_2 < 0,$$

$$a_{33} = \delta\gamma y_2 - d_3.$$

The characteristic equation of $J(E_2)$, may be composed as:

$$\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0 = 0, \tag{5.9}$$

where the coeffcients of Eq.(5.9) stated by:

$$A_2 = -(a_{11} + a_{22} + a_{33}),$$

$$A_1 = a_{11}a_{33} + a_{22}a_{33} + a_{11}a_{22} - a_{12}a_{21},$$

$$A_0 = (a_{12}a_{21} - a_{11}a_{22})a_{33}.$$

Clearly, $a_{33} < 0$ when:

$$y_2 < \frac{d_3}{\delta\gamma}, \tag{5.10}$$

And $(a_{11} + a_{22}) < 0$ when :

$$y_2 < \frac{r}{\beta k p}. \tag{5.11}$$

So, from (5.10) and (5.11), one may obtain $A_2 > 0$ when the condition (5.7) holds. Further, we have $A_0 > 0$ when $(a_{12}a_{21} - a_{11}a_{22}) < 0$, and this happen when (5.6) holds. Moreover, can get

$$\Delta = A_1A_2 - A_3 = (a_{11} + a_{22})\big((a_{12}a_{21} - a_{11}a_{22}) - a_{33}(a_{11} + a_{22} + a_{33})\big) > 0.$$

Hence, according to the Routh-Hurwitz criteria, all the eigenvalues of $J(E_2)$ have negative real parts, if the conditions (5.6) and (5.7) are hold. Therefore, E_2 is locally asymptotically stable. ■

Theorem 5.4: The positive equilibrium point $E_3 = (x_3, y_3, z_3)$ is locally asymptotically stable in R_3^+ , provided that the following condition holds:

$$A > b_{11} > \max\{B, C\}, \tag{5.12}$$

Where,

$$A = \frac{b_{12}b_{21} + b_{23}b_{32}}{b_{22}}, \qquad B = \frac{b_{13}b_{21}}{b_{22}} \ and \ C = \frac{B_0}{B_1} + b_{22}.$$

Here, b_{ij} , $i, j = 1,2,3$ are the elements of matrix (5.1) at E_3, and $B_i, i = 0,1,2$ are the coeffcients of its characteristic equation.

Proof : The Jacobian matrix of system (2.1) at the equilibrium point E_3 is given by:

$$J(E_3) = \begin{bmatrix} -b_{11} & -b_{12} & b_{13} \\ b_{21} & b_{22} & -b_{23} \\ 0 & b_{32} & 0 \end{bmatrix} \tag{5.13}$$

where,

$$b_{11} = \frac{rx_3}{k} \quad , \quad b_{21} = \frac{\beta(m + py_3)y_3}{(1 + ez_3)},$$

$$b_{12} = \frac{(m + 2py_3)(d_2 + \gamma z_3)}{\beta(m + py_3)} \quad , \quad b_{22} = \frac{py_3(d_2 + \gamma z_3)}{(m + py_3)},$$

$$b_{13} = \frac{e(d_2 + \gamma z_3)y_3}{\beta(1 + ez_3)} \quad , \quad b_{23} = (\frac{e(d_2 + \gamma z_3)y_3}{(1 + ez_3)} + \gamma y_3,$$

$$b_{32} = \delta\gamma z_3.$$

Additionally, the characteristic equation of matrix $J(E_3)$ may be written as follows::

$$\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0 = 0, \tag{5.14}$$

where ,

$$B_2 = (b_{11} - b_{22}),$$

$$B_1 = (b_{12}b_{21} + b_{23}b_{32} - b_{11}b_{22}) ,$$

$$B_0 = (b_{11}b_{23} - b_{13}b_{21})b_{32}.$$

Furthermore,

$$\Delta = B_2B_1 - B_0. \tag{5.15}$$

Now, with the help of the specified condition (5.12), the Routh - Hurwitz criteria is satisfied , i.e. $B_i(i = 0,1,2) > 0$ and $\Delta > 0$. As a result, all eigenvalues of $J(E_3)$ have negative real part. It follows that E_3 is locally asymptotically stable. ■

The following theorem looks at the Hopf bifurcation existence requirements for system (2.1).

Theorem 5.5 The system (2.1) in the vicinity of E_3 with regard to p may undergo Hopf bifurcation if:

$$(B_1 - 3\varpi^2)\left(\frac{dB_0}{dp} - \varpi^2\frac{dB_2}{dp}\right) + 2B_2\varpi^2\frac{dB_1}{dp} \neq 0. \tag{5.16}$$

Proof : According to the existence requirements of Hopf bifurcation (see, [54]), it is observed that adynamical system experiences a Hopf bifurcation when the characteristic equation (5.14) has three roots, the first two are purely imaginary $\pm i\varpi$ and the third root has a negative real part and Re $\left(\frac{d\lambda}{dp}\right)_{p=p^*} \neq 0$. Imaginary roots make the equation $\Delta = B_2B_1 - B_0 = 0$ satisfy. To determine if a periodic solution exists, we shall look at the transversality condition. For this differentiating Eq. (5.14) with respect to p, we state:

$$(3\lambda^2 + 2B_2\lambda + B_1)\frac{d\lambda}{dp} + \lambda^2\frac{dB_2}{dp} + \lambda\frac{dB_1}{dp} + \frac{dB_0}{dp} = 0,$$

then can have:

$$\frac{d\lambda}{dp} = -\frac{\lambda^2\frac{dB_2}{dp} + \lambda\frac{dB_1}{dp} + \frac{dB_0}{dp}}{3\lambda^2 + 2B_2\lambda + B_1}.$$

Moreover, for $\lambda = i\varpi$, may have that

$$\left(\frac{d\lambda}{dp}\right)_{\lambda=i\varpi} = -\frac{\left(\frac{dB_0}{dp} - \varpi^2\frac{dB_2}{dp}\right) + i\varpi\frac{dB_1}{dp}}{(B_1 - 3\varpi^2) \quad + 2iB_2\ \varpi},$$

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$$\left(\frac{d\left(Re(\lambda)\right)}{dp}\right)_{\lambda=i\varpi} = \frac{(B_1 - 3\varpi^2)\left(\frac{dB_0}{dp} - \varpi^2\frac{dB_2}{dp}\right) + 2B_2\,\varpi^2\frac{dB_1}{dp}}{(B_1 - 3\varpi^2)^2 + 4B_2^2\varpi^2}$$

So, one may obtain $Re\left(\frac{d\lambda}{dp}\right)_{\lambda=i\varpi} \neq 0$ when the condition (5.16) holds.

■

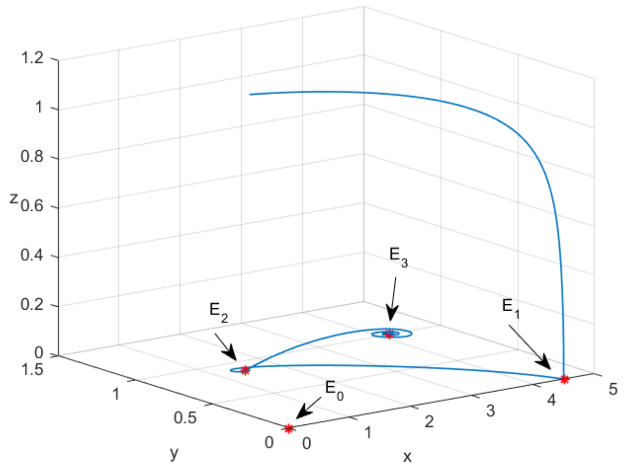

Figure 1: 3D Phase portrait of system (2.1) for data in Table 1.

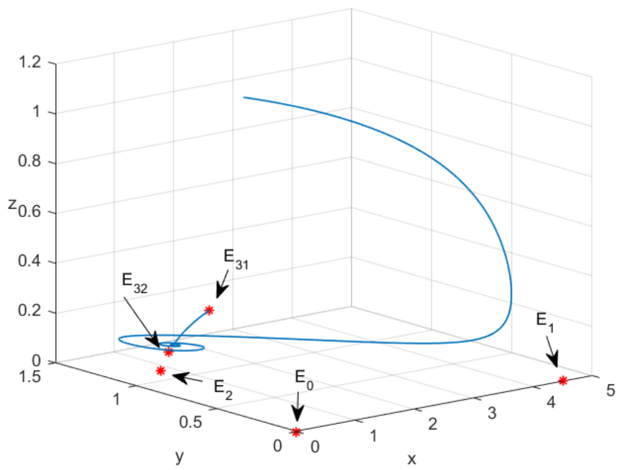

Figure 2: 3D Phase portrait of system (2.1) for data in Table 2.

6. Numerical Simulation

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In this section some numerical simulations have been shown in detail in order to verify the accuracy of the analytical results and make clear the impact of changing the parameters values on the dynamics of the proposed system. On the basis of two sets of hypothetical parametric values for model (2.1) as shown in Tables 1 and 2, and with the aid of Matlab 8.1, numerous phase diagrams have been provided to illustrate the dynamical characteristics of system (2.1), and to depict the theoretical analyses graphically,

Table 1: Parameters values of model (2.1)

r	k	d_1	m	p	e	β	d_2	γ	δ	d_3
1	5.15	0.125	0.25	0.5	2	0.1	0.1	0.3	0.5	0.09

Table 2: Parameters values of model (2.1)

r	k	d_1	m	p	e	β	d_2	γ	δ	d_3
1	5.15	0.125	0.3	0.3	2	0.2	0.1	0.3	0.5	0.175

Now in order to verify the existence and stability of the positive equilibrium $E_3(x_3, y_3, z_3)$, we used the above parameter values with $(x(0), y(0), z(0)) = (2.5,1.25.1)$.

At first, with the values of the parameters given in Table 1, we noticed that: the solution of system (2.1) has $y_3 = 0.6$, the condition (4.17) is true, and $s_4 < 0$. Additionally, Eq.(4.16) has three real roots given by $z_3 = -1.1513, z_3 = -0.3093$, and $z_3 = 0.1272$. Hence system (2.1) has a unique positive equilibrium point $E_3 = (3.1515,0.6,0.1272)$, and three boundary equilibrium points $E_0 = (0,0,0), E_1 = (4.5063,0,0), E_2 = (1.4650,0.8651,0)$. Moreover, at E_3, the condition of theorem (5.4) is satisfied as $b_{11} = 0.6119, A = 0.8244, B = 0.1114, C = 0.2616$, also, $J(E_3)$ has three eigenvalues with negative real parts given by $\lambda_1 = -0.5168, \lambda_2 = -0.0099 + 0.0753i, \lambda_3 = -0.0099 - 0.0753i$, and consequently E_3 is asymptotically stable as plotted in Figure 1.

Secondly, when using the parameter values shown in Table 2, we noticed that: the solution of system (2.1) has $y_3 = 1.1667$, the condition (4.17) is not true, the conditions (4.18)-(4.19) hold, $s_3 = -0.4716 < 0$, and $s_4 = 0.0219 > 0$. Additionally, Eq.(4.16) has three real roots given by $z_3 = -1.5881, z_3 = 0.1962$ and $z_3 = 0.0586$. Hence system (2.1) has two positive equilibrium points $E_{31} = (1.013,1.1667,0.0586), E_{32} = (1.7013,1.1667,0.1962)$, and three boundary equilibrium points $E_0 = (0,0,0), E_1 = (4.5063,0,0), E_2 = (0.7821,1.1311,0)$, (see Figure 2). In this case we noted that, at E_{32} the condition (5.12) holds by $b_{11} = 03303$, $A = 1.7681, B = 0.2353$ *and* $C = 0.0996$. Hence, the Jacobian matrix (5.13) at E_{32} has three eigenvalues with negative real parts given by $\lambda_1 = -0.0144, \lambda_2 =$

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$-0.1152 + 0.3262i$, $\lambda_3 = -0.1152 - 0.3262i$, and consequently E_{32} is asymptotically stable as plotted in Figure 2. While at E_{31}, matrix (5.13) has real positive eigenvalue $\lambda_1 = 0.0053$, and two eigenvalues with negative real parts $\lambda_2 = -0.0579 + 0.3629i$ and $\lambda_3 = -0.0579 - 0.3629i$. Therefore, E_{31} is saddle point for system (2.1).

For checking the existence and stability of the top predator-free equilibrium point E_2, we used the hypothetical data set given in Table 2 in two cases. First, by reducing thevalue of β to 0.15, we can see that $k = 2.2222$ less than $\frac{k(r-d_1)}{r} = 4.5063$, and hence the existence condition (4.8) mets. Consequently, system (2.1) has unique top predator-free equilibrium point $E_2 = (1.0723,1.0725,0)$. Also, for applied theorem (5.3), we possess that: $\frac{k(m+py2)(m+2py2)}{rp} = 10.0699, \frac{d_3}{\delta\gamma} = 1.1667$ and $\frac{r}{\beta kp} = 4.315$. This implies the conditions (5.6) and (5.7) are satisfied, and the Jacobian matrix (5.8) earns the eigenvalues $\lambda_1 = -0.0141, \lambda_2 = -0.0782 + 0.2903i$, and $\lambda_3 = -0.0782 - 0.2903i$. Therefore, the system (2.1) has the asymptotically stable equilibrium point $E_2 = (1.0723,1.0725,0)$, as plotted in figs. 3 and 4. In case 2, by reducing the values of m and β to 0.215 and 0.15 at respectively, yield $k = 4.6512$ greater than $\frac{k(r-d_1)}{r} = 4.5063$ and p greater that $\frac{m^2}{r-d_1} = 0.0528$. This demonstrates that the existence conditions (4.9) and (4.10) met. Consequently, system (2.1) has two top predator-free equilibrium points $E_{21} = (4.4735,0.0285,0)$ and $E_{22} = (1.9754, 0.9708,0)$. Once again, from applying theorem (5.3), it follow that: $\frac{d_3}{\delta\gamma} = 1.1667$ and $\frac{r}{\beta kp} = 6.4725$, and at $E_{12}, \frac{k(m+py2)(m+2py2)}{rp} = 0.8906$. This indicates that the conditions (5.6) is not satisfied. Since $A_0 < 0$ is present in this situation, which leads Matrix (5.8) to has three eigenvalue $\lambda_3 = 0.0031, \lambda_2 = -0.8679$,and $\lambda_3 = -0.1707$, one of which is positive. But, by at E_{22} the following results are obtained: $\frac{k(m+py2)(m+2py2)}{rp} = 6.9301$, and hence the conditions (5.6) and (5.7) are met. As a result, the matrix (5.8) has three eigenvalue $\lambda_1 = -0.0294, \lambda_2 = -0.1630 + 0.1696i$, and $\lambda_3 = -0.1630 - 0.1696i$, all of which have negative real parts. Therefore, the system (2.1) has saddle point E_{21} and asymptotically stable equilibrium point E_{22}, as shown in figs.5 and 6.

Now, for confirming the existence and stability of predator-free equilibrium point E_1, we used $m = 0.2$ and the other parameter values as given in the data set of Table 2. It was noted that the condition (4.1) has been satisfied because $x_1 = 4.5063$ and $k = 2.2222$. Next, we can state that the system (2.1) possesses the asymptotically stable point $E_1 = (4.5063,0,0)$ and that theorem (5.2) has been applied. Figure 7 shows that the solution of system (2.1) x,y, and z converge to their steady state solutions E_1. Moreover, with the help of

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Theorem (5.2) we indicate that the system (2.1) at E_0 becomes unstable when $r > d_1$, which is obviously shown in Figs. 1-7, and stable asymptotically when $r < d_1$ as displayed in Fig.8.

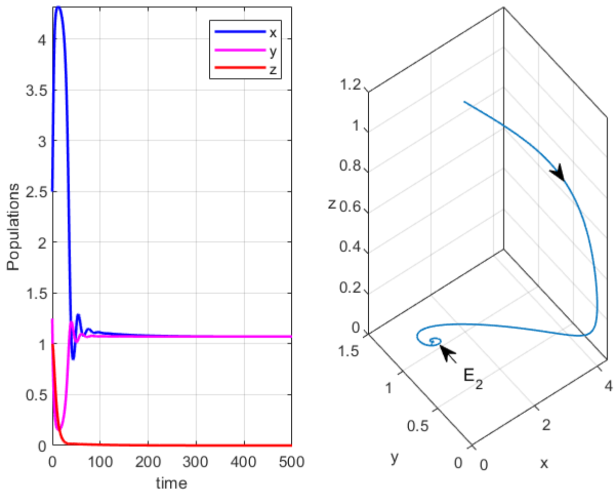

Figure 3: Time series and 3D Phase portrait of system (2.1) for data in Table 2 with $\beta = 0.15$.

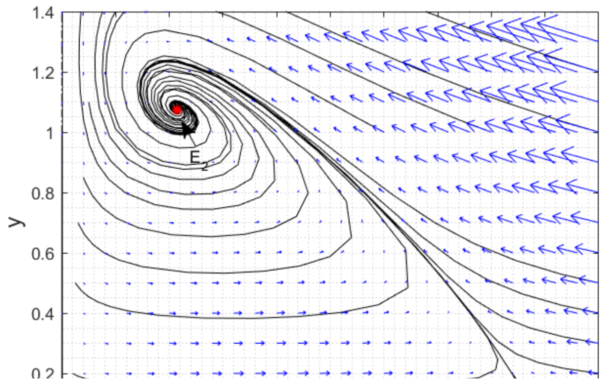

Figure 4: 2D Phase portrait of system (2.1) for data in Table 2 with $\beta = 0.15$.

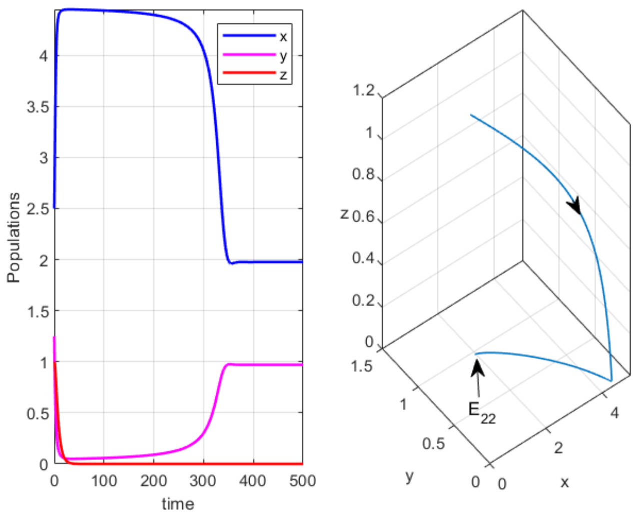

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Figure 5: Time series and 3D Phase portrait of system (2.1) for data in Table 2 with $m = 0.15, \beta = 0.15$.

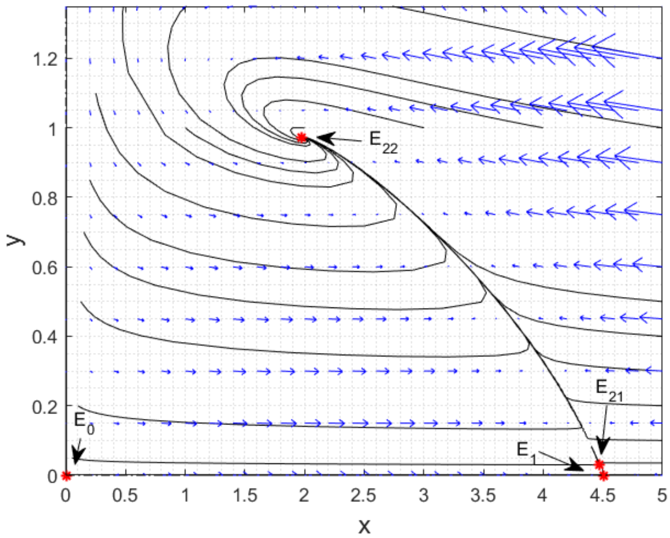

Figure 6: 2D Phase portrait of system (2.1) for data in Table 2 with $m = 0.15, \beta = 0.15$.

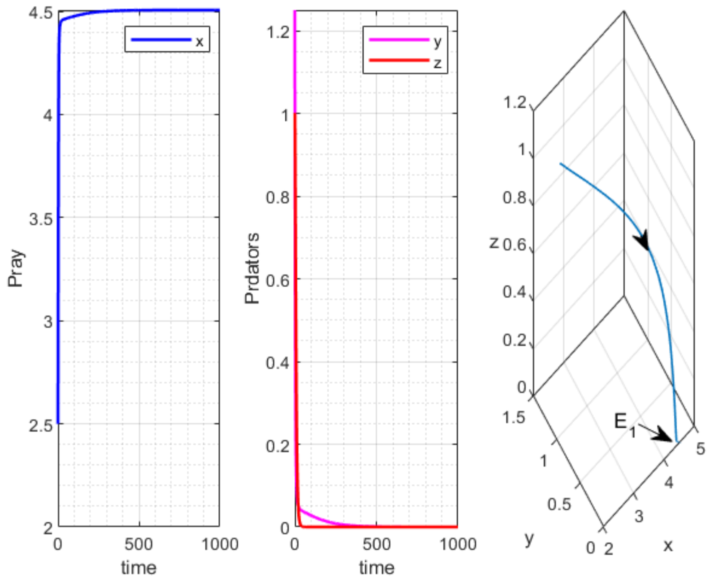

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Figure 7: Time series and 3D Phase portrait of system (2.1) for data in Table 2 with $m = 0.2$.

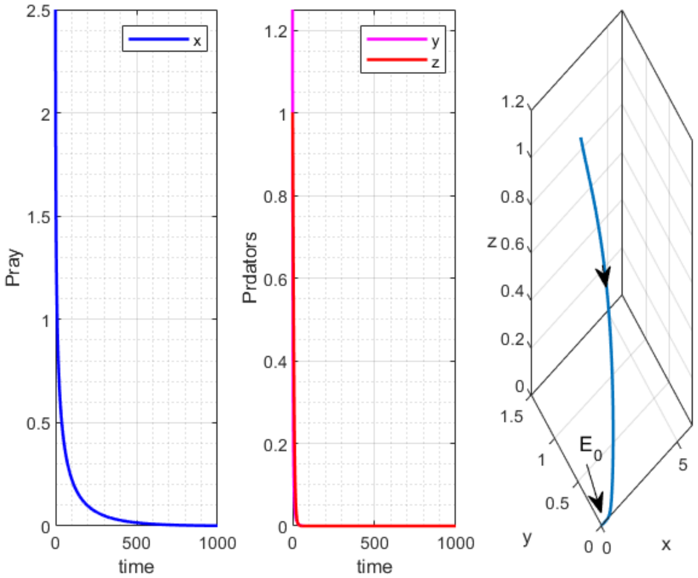

Figure 8: Time series and 3D Phase portrait of system (2.1) for data in Table 2 with r = 0.12.

On the other hand, we observed that when p takes as a bifurcation parameter, the system may exhibit a Hopf bifurcation. Figure 9 plots the phase portrait for several values of p to demonstrate this occurrence. The system exhibits stable positive steady-state behavior for $p = 0.05$ and $p = 0.2$, as seen in the first and last subplots, while the model exhibits a stable periodic behavior for $p = 0.1$ and $p = 0.15$ as seen in the middle subplots.

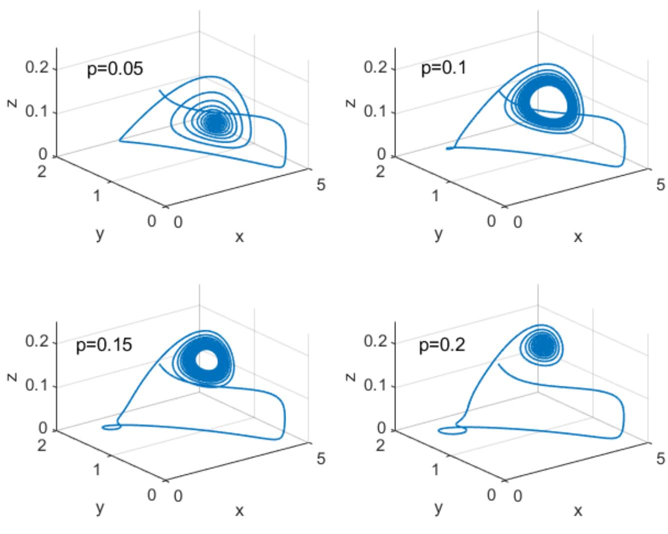

Figure 9: 3D Phase portrait of system (2.1) for data used in Figure case2 with $m = 0.14, d_3 = 0.15$ and $p = 0.05,0.1,0.15,0.2$.

To demonstrate the combined effect of e and p on the dynamic of system (2.1), we have drawn Figures 10 and 11 in two levels of fear effect, $e = 1.5,5$, with three values of hunting cooperation p = 0.05, 0.55, and 0.9. We used the value $k = 5.125$ in Figure 1, and we see that: at fix $e = 1.5$ and gradually increasing the hunting cooperation effect, the solutions still converge asymptotically to positive equilibrium point E_3(see top subfigures of time series and phase portrait). If e = 5 is raised, the model (2.1) continues to show stable oexistence at p = 0.05 and displays periodic limit cycle dynamics as p rises (see bottom subfigures of time series and phase portrait). Figure 11 demonstrates that with k = 3.75 the dynamics of system (2.1) has behavior as in Figure 10 for e = 1.5 or p = 0.05,0.55, but at (e,p) = (5,0.9) the dynamics behavior shifts back to stable E_3.

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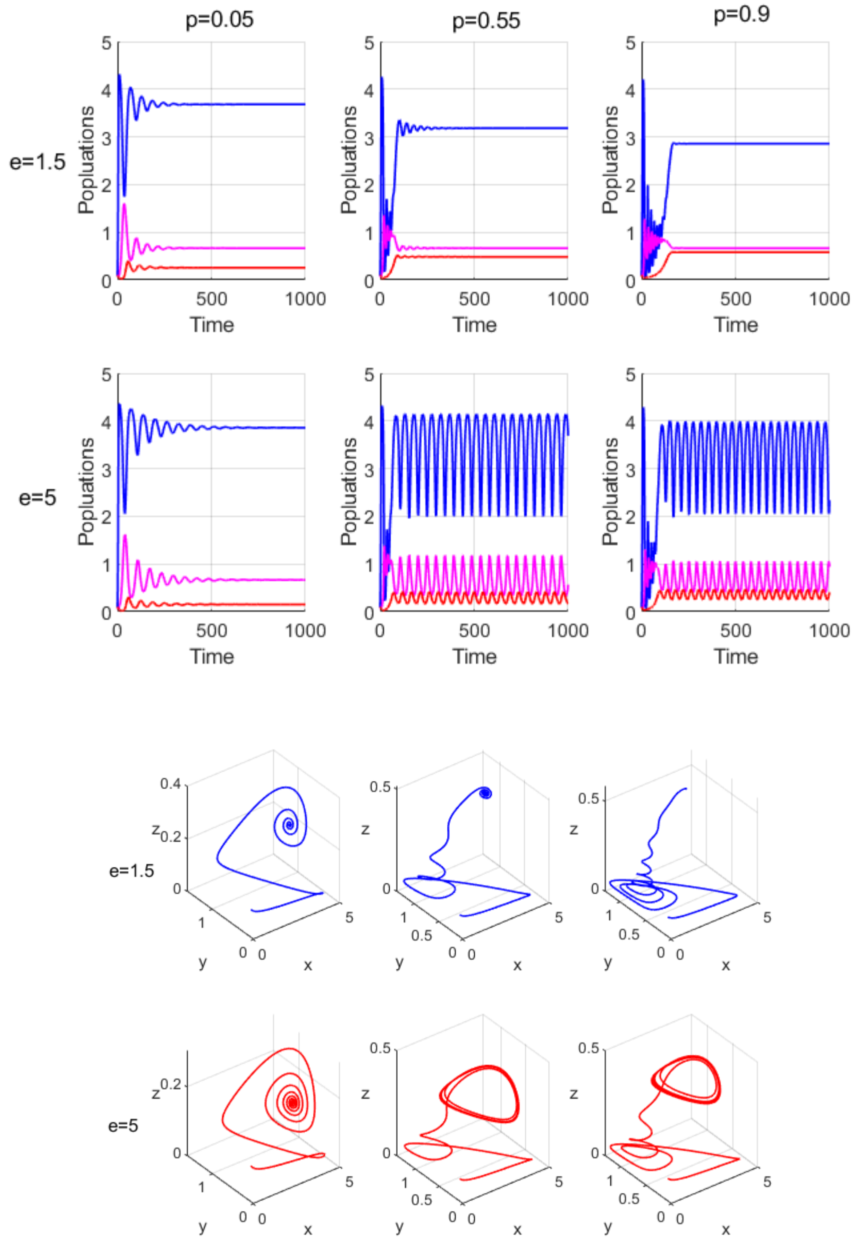

Figure 10: Time series and 3D Phase portrait of system (2.1) for data are taken from Table 2 with d_3 = 1, e = 1.5,5 and, p = 0.05,0.5,0.9.

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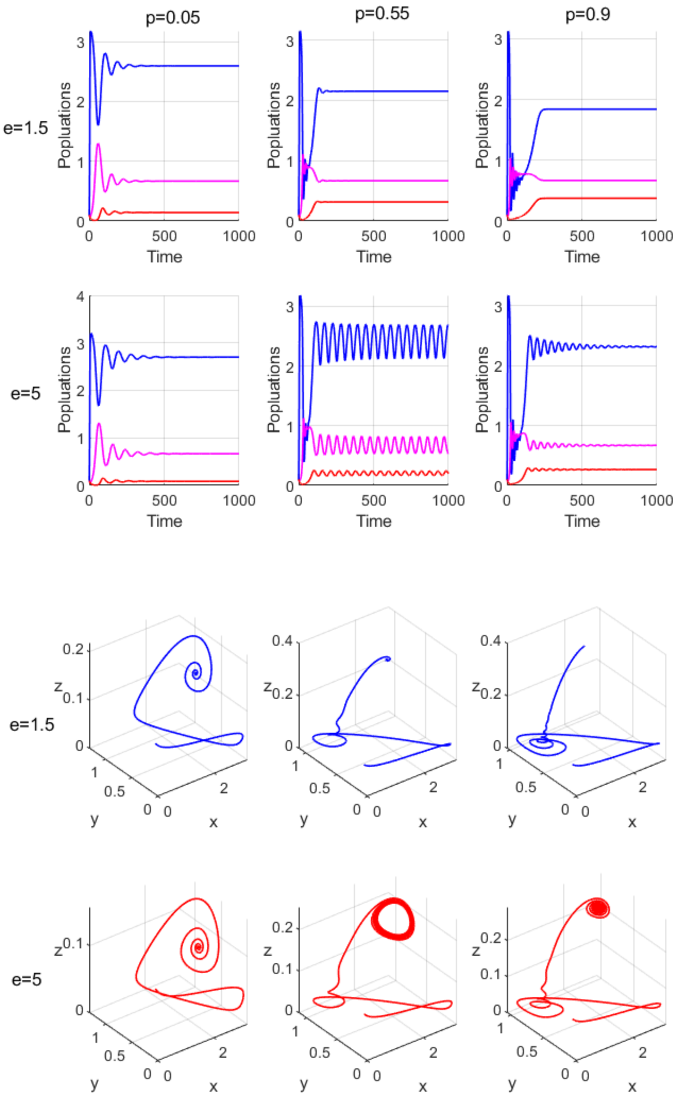

Figure 11: Time series and 3D Phase portrait of system (2.1) for data are used in Figure 10 with k = 3.75.

It signiffes that the system exhibits limit cycle oscillation at large values of the fear effect and hunting cooperation parameters. At small values of k, this oscillation may be regulated by increasing the value of the hunting cooperation parameter.

7. Conclusion

The study of prey-predator models is crucial because it has implications for conservation biology. Models of the three-level food chain take into account fear in populations of middle predators and/or prey. In the current study, we solely consider a food chain predator-prey system with fear in the middle predator population. We also take into account the fact that the middle predator population cooperates in hunting. The objective is to investigate the effect of induced fear and cooperation between lower predators on the dynamic behavior of this system. The properties of the solution of the model (such as positivity and boundedness) are discussed. From the theoretical analysis of the model, it is found that the system (2.1) consists of four biologically valid equilibrium points. The trivial equilibrium point always exists but simultaneous extinction of all the populations cannot be possible if the prey survival condition is valid. The predators-free equilibrium point always exists when the prey survival condition is met, but simultaneous extinction of all the predators cannot be possible when the prey density more than $\mathcal{K}$. While the existence of E_2 and E_3 is constrained by parameters related to the cooperation of hunters and the fear effect with other parameters, their stability analysis has been explored by linearization approach, and the necessary conditions are derived. Also, It is observed that hunting cooperating in the middle predator may produces two co-existence equilibria. One positive equilibrium point is locally asymptotically stable depending on the values of p and e, whereas another positive equilibrium point is unstable. Finally, numerical simulations are given to exemplify the effcacy of theoretical results. It is demonstrated that the dynamical behavior of system (2.1) exhibits a variety of dynamics, without becoming chaotic, and that the presence of cooperation and fear have a significant impact on the stability of the dynamics. Observe that the presence of the fear effect in the intermediary predator might ensure the prey's and middle predator's survival, but may also raise the possibility of top predator extinction. Hence, a high amount of fear has a bad impact on the top predator. Nevertheless, cooperation between middle predators may be guaranteed the survival of top predators.

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