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The LaPlace Homotopy analysis method for solving systems of fractional integral differential equations

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Abstract

The paper presents the Laplace homotopy analysis method (LHAM) as an efficient and robust approach to solving systems of fractional integral differential equations (FIDE). The combination of the Laplace transform and the homotopy analysis method (HAM) solves the convergence and computational problems that often occur when solving fractional systems. By transforming differential equations into algebraic ones, LHAM increases the ease of working with partial derivatives while maintaining flexibility and stability using auxiliary parameters. The paper demonstrates the effectiveness of LHAM through examples, comparing its performance with established methods such as the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM), showing its superior accuracy and efficiency.

Keywords

Laplace Homotopy Analysis Method (LHAM) , Fractional Integral Differential Equations (FIDEs) , Laplace Transform , Homotopy Analysis Method (HAM), Fractional Calculus , Stability

Introduction

Within the last few decades, fractional-order integral differential equations have gained significant attention because of their widespread applications to various scientific and engineering fields, such as viscoelasticity, control theory, fluid mechanics, and biological systems**(Sun et al., 2019)**. These equations can model more adequately many complex processes, featuring memory and hereditary properties, by means of their derivatives of arbitrary order**(Yates and modelling, 1994)**. However, once these are set in system forms, their solution presents considerable analytical and computational challenges. Conventional techniques include the Adomian Decomposition Method, the Variational Iteration Method, and the Homotopy Perturbation Method for such problems. Despite the usefulness of these techniques, they are frequently plagued by disadvantages that relate to

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convergence and laborious computations, especially for nonlinear and coupled systems**(Al Baghdadi et al., 2024)**.

In light of these challenges, this paper would recommend the LHAM as one of the most robust and efficient tools toward obtaining an approximate solution for systems of fractional integral differential equations. By virtue of its linearity properties, the Laplace transform is a well-known powerful tool in the transformation of a differential equation into an algebraic equation. Meanwhile, the Homotopy Analysis Method guarantees convergence with flexibility because it introduces auxiliary parameters and functions.

. The integration of these two methods, namely LHAM, develops the strength of each method for an effective approach to handle the complexities arising in fractional systems. LHAM not only facilitates the analytical handling of fractional derivatives but also enhances the convergence and stability criteria of the solution. The present research has aimed to show the application of LHAM to different systems of FIDEs with efficiency. The paper also describes how LHAM can be used to find the exact solution of such complicated fractional systems by the detailed analysis along with numerical examples. A comparison of the results obtained by this technique with already developed techniques can be made and advantages of computational simplicity and accuracy with LHAM are presented. The obtained results contribute to the continuing work in the area of fractional calculus and give a powerful technique to analysts and practitioners studying systems with dynamics described by fractional operators.

Background

Fractional calculus is a branch of mathematics that deals with integra-tions and derivations of non-integer order. Recently it has obtained considerable attention because it enables, at least in some cases, more accurate modeling of real-world problems than traditional integer-order calculus**(Daftardar-Gejji, 2013)**. The FDE systems possess the properties of memory and hereditary effects which are of primary importance in modeling complex processes of physics, engineering, biology, and finance**(Kolmanovskii and Myshkis, 2012)**. For example, within the framework of fractional models, one can describe such phenomena as the viscoelastic materials, anomalous diffusion, and biological systems where the processes depend on not only the current state, but also on the history of the system. However, the involved mathematical intricacy with fractional derivatives makes the finding of an exact solution of such an equation quite difficult**(Nisar et al., 2024)**.

Traditional analytical techniques for the solution of differential equations-like the classical Laplace transform, Fourier transform, and method of separation of variables-prove to be inadequate in dealing with such fractional systems owing to their very complicated nature**(Adkins and Davidson, 2012)**. Several numerical and semi-analytical techniques are thus developed. Fractional differential equations have been solved using methods like the Adomian Decomposition Method, the Variational Iteration Method, and the Homotopy Perturbation Method**(Chakraverty et al., 2019)**. Despite their apparent promise for approximate solutions, these methods generally suffer from a variety of drawbacks related to convergence, accuracy, and nonlinearities treatment. These issues get further complicated in the case of systems of fractional integral differential equations because of the interplay among many such equations, introducing their own further complications**(Baleanu et al., 2023)**.

Here, the LHAM becomes an important alternative through combining the Laplace transform and Homotopy Analysis Method. The Laplace transform will be more convenient in changing the fractional differential equations to algebraic ones in the Laplace domain, since it becomes easier to handle fractional differential operators. On the other hand, HAM represents a flexible framework; it allows control of convergence and treatment of nonlinearities via construction of homotopies**(Schiff, 2013)**. Coupling these two approaches, LHAM will present an influential approach toward obtaining analytical and semi-analytical solutions to systems of fractional differential equations. It serves as the background for bringing out the study on the efficacy of LHAM in solving complicated fractional systems and sheds new light into the application of fractional calculus.

Preliminaries and Basic Concepts

Fractional Calculus

Fractional calculus extends the conventional notion of differentiation and integration to non-integer orders, thus allowing much more flexibility in modeling systems with memory and hereditary properties. That is, whereas integer-order calculus only considers the rate of change at an instant, fractional calculus takes into consideration the entire prior history of the function**(Valentim et al., 2021)**. For this reason, often it turns out to be particularly suitable when describing those processes showing anomalous diffusion, viscoelastic behavior, and other complex dynamics. The definitions of fractional derivatives given in literature and used most frequently are the Riemann-Liouville and Caputo derivatives**(Goychuk and Physics, 2009)**.

The **Riemann-Liouville fractional derivative** of order $\alpha > 0$ of a function f (t) is defined as:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{t(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$

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where n = ⌈α⌉ is the smallest integer greater than or equal to α and Γ(·) denotes the Gamma function. This definition extends the n-th order derivative due to the involvement of a convolution of the function f(t) with a power-law kernel 1/((t-τ)a-n+1) which therefore brings in the memory of f(t). The Riemann-Liouville derivative is especially applicable in modeling physical processes where the initial state of the system is of relevance.

where α is the order of the Caputo fractional derivative.:

$$^{c}D_t^{a} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{a-n+1}} d\tau,$$

where f(n)(τ) represents the n-th derivative of f(τ). The Caputo fractional derivative differs from the Riemann-Liouville fractional derivative because in the Caputo derivative the n-th derivative of the function inside the integral appears. Such a property makes the Caputo derivative more suitable in view of the initial value problems since in applications it will be possible to prescribe the traditional integer order initial conditions directly, which often may be more convenient for physical and engineering applications.

A simple example of a fractional differential equation with the Caputo derivative can be given as:

$$^{c}D_t^{0.5} y(t) + y(t) = t^2 \, , \; y(0) = 0$$

To solve this equation, we use methods such as the Laplace transform, which can handle fractional derivatives efficiently by converting them into algebraic equations.

Laplace Transform

The most powerful integral transformation of time functions into functions of the complex variable s is the Laplace transformation. For a function f(t), the Laplace transform is defined as:**(LePage, 2012)**

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The Laplace transform has many properties, all of which combined allow the Laplace transform to be an extremely powerful tool for the solution of differential equations. Some of these include: linearity, differentiation property, and convolution property**(Schiff, 2013)**. Of all of the important properties of the Laplace transform perhaps that which makes the Laplace transform most useful is that it can reduce derivatives to an algebraic expression in the s-domain. The nth derivative of f(t) is defined as follows:

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$$L\left\{\frac{d^n f(t)}{dt^n}\right\}=s^n F(S)-s^{n-1} f(0)-s^{n-2} f'(0)-\dots-f^{(n-1)}(0)$$

By considering fractional derivatives, it is possible to extend the Laplace transform to a noninteger order. The Laplace transform of the Caputo fractional derivative of order α of a function $f(t)$ is given by

$$\mathbf{L}\{{}^{C}D_t^{\alpha} f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

where $n = \lceil\alpha\rceil$. This property considerably simplifies the process for solving fractional differential equations since it reduces the task of finding an algebraic equation in the s-domain, manipulating, and then inverting back to the time domain.

Now, to apply the Laplace transform in order to solve a fractional differential equation consider the following example:

${}^{c}D_t^{0.5} y(t)+y(t)=t^2 \quad ,y(0)=0$

Applying the Laplace transform to both sides of this equation results in:

$s^{0.5} Y(s)+Y(s)=2/s^3$

Here, $Y(s) = L\{y(t)\}$. We can take out the common factor $Y(s)$:

$Y(s)=2/(s^3 (s^{0.5}+1))$

Where $Y(s)= 2/(s^3*(s^{0.5}+1))$.

To find $y(t)$, we must take the inverse Laplace transform of $Y(s)$. Although the answer may not be in simple closed form, it will, in most instances, be expressible in terms of known functions or be calculable numerically. This example shows just one direction in which the Laplace transform provides a powerful way of dealing with fractional derivatives, in that it reduces the problem to an algebraic equation.

From the perspective of a Laplace transformation with respect to fractional integral differential equations, what truly matters is how the transformation changes such fractional derivatives into algebraic terms. By combining techniques such as the Homotopy Analysis Method and using a Laplace transform, one can give an analytical framework of solution for complicated systems of fractional equations. The main purpose is the inspiration of an analytical technique which is capable of handling the inherent complications of the fractional-order differential systems with its excellent capability inherited from both Laplace transformation and HAM.

By LHAM, the nonlinearity and memory effect of fractional systems can be treated systematically to find an approximate or exact solution. In the forthcoming sections, how to apply LHAM to different kinds of systems of fractional integral differential equations will be deduced in order to explore its possibility in acting as a powerful tool in fractional calculus.

Homotopy Analysis Method (HAM)

The Homotopy Analysis Method represents an analytic procedure in frames of highly nonlinear differential equations. It is also the important complement to the traditional perturbation methods**(Dyke and Dyke, 2001)**. Unlike the perturbation methods based on small parameters which expand the solutions, HAM provides a systematic and parameter-independent approach for series solution**(Plyasunov and Arkin, 2007)**. The crucial philosophy behind HAM is to build a homotopy-continuous transformation from a simple, easily analyzed problem into the original, usually more complicated one**(Liu, 2017)**. This transformation really enables HAM to transform a complex nonlinear problem into a series of simpler problems which are thus easier to be solved analytically. In HAM, the homotopy is constructed through an auxiliary linear operator L, an auxiliary function H(τ),and embedding parameter p. A general nonlinear differential equation can be written as N[u(t)] = 0, where N is a nonlinear operator, and u(t) is the unknown function. Following HAM, we first select an initial approximation u0(t) for the solution and an auxiliary linear operator L such that L[u0(t)] = 0. We next construct a homotopy continuously deforming the linear operator L into the original nonlinear operator N. **(Shukla et al., 2012)**

This is formulated as:

$$(1-P)L[\emptyset(t,p)]+PH(t)N[\emptyset(t,p)]=0,$$

where ϕ(t; p) is a family of functions depending on the embedding parameter p ∈ [0, 1]. For p = 0, the homotopy equation reduces to the linear problem L[ϕ(t; 0)] = 0, for which an exact solution is known. If p = 1, then it becomes the original nonlinear problem N [ϕ(t; 1)] = 0.

The auxiliary function H(t) and the linear operator L give flexibility to the method in order to handle the convergence region and the rate of convergence. One of the unique features of HAM is the so-called convergence-control parameter $\hbar$. This does not exist in

the other analytical approaches and allows one to make an adjustment in such a way that series solution can be converged. In p, u(t) was represented as a series power:

$$\phi(t,p)=u_0(t)+\sum_{m=1}^{\infty}u_m(t)P^m,$$

where each um(t) can be determined iteratively. Setting p = 1 gives the approximate solution to the original problem:

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t)$$

This series can be truncated to obtain an approximate analytical solution. The convergence-control parameter $\hbar$ and the auxiliary function H(t) play crucial roles in ensuring that the series converges and provides an accurate solution.

To illustrate HAM, consider the nonlinear differential equation:

$$\frac{du(t)}{dt} + u(t)^2 = 0, \qquad u(0) = 1$$

Here, N [u(t)] = $\frac{du(t)}{dt} + u(t)^2$. We choose L as a simple linear operator L[u(t)] = $\frac{du(t)}{dt}$ and an initial guess u0(t) = 1. Constructing the homotopy and applying HAM, we iteratively find the terms um(t), leading to an approximate series solution.

Laplace Homotopy Analysis Method (LHAM)

LHAM extended the standard HAM by embedding the Laplace transform into the homotopy framework. With the strengths of the Laplace transform and the HAM incorporated in it, LHAM is a versatile method that tries to solve a wide range of fractional integral differential equations, especially those involving complex nonlinearities with fractional derivatives**(Veeresha et al., 2019)**. The Laplace transform reduces the differential equations to algebraic equations, while HAM provides a systematic approach to handle nonlinearities analytically. LHAM takes advantage of the above two methods for a more effective solution technique**(Bonkile et al., 2018)**.

First of all, to apply LHAM the original fractional differential equation should be transformed by the Laplace transform. Let the following form be a representative for the fractional differential equation:

$${}^{c}D_t^{a} u(t) + N[u(t)] = g(t), \;\; 0 < a < 1,$$

where N is a nonlinear operator, and g(t) is a source term. Applying the Laplace transform, we convert this equation into the s-domain:

$$s^a u(s) - s^{a-1} u(0) + L\{N[u(t)]\} = L\{g(t)\}$$

Letting u(0) = u0, we rearrange the equation to isolate U (s):

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$$U(s) = \frac{s^{\alpha-1}u_0 + L\{g(t)\} - L\{N[u(t)]\}}{s^{\alpha}}$$

This algebraic expression forms the basis for constructing the homotopy. In LHAM, we construct a homotopy in the s-domain as follows:

$$(1-P)L[\phi(s;p)] + PH(s)[s^{a}\phi(s;p) - s^{a-1}u_0 - L\{g(t)\} + L\{N[\phi(t;p)]\}] = 0$$

where Φ(s; p) is the family of functions depending on the embedding parameter p. In this way, by setting p = 0 the equation will reduce to a simple linear problem whereas for p = 1 it will represent the original fractional differential equation in the s-domain.

Using the properties of the Laplace transform, the series solution terms can be iteratively computed. The obtained s-domain series is inverted back to time domain using the inverse Laplace transform to get the approximate solution u(t). This convergence-control parameter $\hbar$ and the auxiliary function H(s) give flexibility in ensuring the convergence of the

Solution Series. Consider as an example, the fractional differential equation:

${}^{c}D_t^{0.5}u(t) + u^2(t) = t,\ \ u(0) = 0$

Applying the Laplace transform, we get:

$$s^{0.5}U(s) + L\{u^2(t)\} = \frac{1}{s^2}$$

To construct the homotopy in s-domain, we iterate for the terms of U(s) by applying LHAM. Taking the inverse Laplace transform results in an approximate solution in the time domain. LHAM greatly enhances dealing with FDEs due to the fact that algebraic simplicity is gained due to the use of the Laplace transform and the capability of HAM to handle nonlinearities. It gives a very powerful means for the analysis of the complicated dynamic systems, as those involved in engineering, physics, and other applied sciences when fractional calculus plays an important role.

Methodology

Formulation of the Problem

The primary objective of this research is to solve a system of fractional integral differential equations using the Laplace Homotopy Analysis Method (LHAM). Consider a general system of fractional integral differential equations expressed in the form:

$$\int cD_t^{a_1}u_1(t) + N_1\left(u_1(t), u_2(t), \dots, u_n(t)\right) = g_1(t), \quad 0 < a_1 < 1,$$

$$\int cD_t^{a_2}u_2(t) + N_2\left(u_1(t), u_2(t), \ldots, u_n(t)\right) = g_2(t), \quad 0 < a_2 < 1,$$

$$\int cD_t^{a_n}u_n(t) + N_n\left(u_1(t), u_2(t), \ldots, u_n(t)\right) = g_n(t), \quad 0 < a_n < 1,$$

Where ${}^{C}D^{ai}$ denotes the Caputo fractional derivative of order αi for the function ui(t), Ni are nonlinear operators acting on u1(t), u2(t), … , un(t), and gi(t) are known source functions. The initial conditions for this system are typically given as:

$$u_i(0) = u_{i0}, \quad i = 1,2, \ldots, n$$

The challenge lies in the fractional nature of the derivatives and the nonlinear interdependencies among the equations, which make standard analytical methods inapplicable. To address this, we apply the Laplace Homotopy Analysis Method (LHAM).

Application of LHAM

Therefore, the major steps which the Laplace Homotopy Analysis Method encompasses are as follows:

1. Laplace Transform: Take the necessary Laplace transformation for each of the involved equations in the system to transform the fractional differential equations to algebraic ones within the frame of the Laplace domain.

2. Construction of Homotopy: Construct a homotopy corresponding to each equation by introducing an embedding parameter p which may continuously deform a simple problem into the original complicated one.

3.iлiufreqiliary Parameters: Introduce the auxiliary parameters such as the convergence-control parameter $\hbar$ which guarantee the convergence of the series solution.

4.ฦษฎIterative Solution: The resulting system has to be solved iteratively in the Laplace domain, after which the inverse Laplace transform is applied to get the solution in the time domain.

Step 1: Laplace Transform

Applying the Laplace transform to the system of fractional differential equations and using the property of the Laplace transform for the Caputo derivative, we have:

$L\{{}^{c}D_t^{ai}u_i(t)\} = s^{ai}U_i(s) - s^{a_i-1}u_i(0),$

where Ui(s) = L{ui(t)} is the Laplace transform of ui(t). Applying this to the system, we get:

$$\int s^{a_1}U_1(s) - s^{a_1-1}u_{10} + L\{N_1(u_1(t), u_2(t), \dots, u_n(t))\} = L\{g_1(t)\},$$

$$\int s^{a_2}U_2(s) - s^{a_2-1}u_{20} + L\{N_2(u_1(t), u_2(t), \dots, u_n(t))\} = L\{g_2(t)\},$$

$$\int s^{a_n}U_n(s) - s^{a_n-1}u_{n0} + L\{N_n(u_1(t), u_2(t), \dots, u_n(t))\} = L\{g_n(t)\},$$

Isolating the unknowns Ui(s):

$$\int U_1(s) = \frac{s^{a_1-1}\ u_{10+L\{g_1(t)\}-L\{N_1(u_1(t),u_2(t),\dots,u_n(t))\}}}{s^{a_1}},$$

$$\int U_2(s) = \frac{s^{a_2-1}\ u_{20+L\{g_2(t)\}-L\{N_2(u_1(t),u_2(t),\dots,u_n(t))\}}}{s^{a_2}},$$

$$\int U_n(s) = \frac{s^{a_n-1}\ u_{n0+L\{g_n(t)\}-L\{N_n(u_1(t),u_2(t),\dots,u_n(t))\}}}{s^{a_n}},$$

Step 2: Construct the Homotopy

We construct a homotopy for each equation in the Laplace domain, introducing the embedding parameter p that ranges from 0 to 1:

$$(1-p)L[\phi_1(s;p)] + PH_1(s)[s^{a_1}\phi_1(s;p) - s^{a_1-1}u_{10} - L\{g_1(t)\} + L\{N_1(\phi_1(t,p), \phi_1(t,p), \dots, \phi\}]$$

$$(1-p)L[\phi_2(s;p)] + PH_2(s)[s^{a_2}\phi_2(s;p) - s^{a_2-1}u_{20} - L\{g_2(t)\} + L\{N_2(\phi_1(t,p), \phi_2(t,p), \dots, \phi\}]$$

$$(1-p)L[\phi_n(s;p)] + PH_n(s)[s^{a_n}\phi_n(s;p) - s^{a_n-1}u_{n0} - L\{g_n(t)\} + L\{N_n(\phi_1(t,p), \phi_2(t,p), \dots, \phi\}]$$

Here, Φi(s; p) is a family of functions that depends on the embedding parameter p, L is a linear operator, and Hi(s) are auxiliary functions. When p = 0, the homotopy corresponds to a simple, linear problem, and when p = 1, it represents the original nonlinear system.

Step 3: Selection of Auxiliary Parameters

We introduce the convergence-control parameter $\hbar$ to guarantee the convergence of the series solution. The auxiliary functions Hi(s) and the linear operator L are chosen such that the iterative solution can be easily procured. Very often the linear operator L is taken to be the identity operator for simplicity, and the auxiliary functions Hi(s) will be chosen in a way depending on the problem under consideration in order to handle the homotopy path effectively.

Step 4: Iterative Solution

By expanding Φi(s; p) as a power series in p:

$$\phi_i(s;p) = \phi_{i0}(s) + \sum_{m=1}^{\infty} \phi_{im}(s)p^m,$$

we obtain a sequence of linear equations for the terms Φim(s). Each Φim(s) is computed iteratively using the recursive relations derived from the homotopy. Setting p = 1, we approximate Ui(s):

$$U_i(s) = \phi_{i0}(s) + \sum_{m=1}^{\infty} \phi_{im}(s)$$

Step 5: Inverse Laplace Transform

Finally, the approximate solutions ui(t) are obtained by taking the inverse Laplace transform of Ui(s):

$$u_i(t) \;=\; L^{-1}\{u_i(s)\}.$$

The resulting series will be an analytical approximation to the original system of fractional integral differential equations, where convergence is guaranteed by the proper choice of the convergence-control parameter $\hbar$.

The approach considers the solution of complicated systems of fractional integral differential equations in a systematic way, using the synergy between the Laplace transform and Homotopy Analysis Method.

Examples and Applications

Worked Example 1: Fractional Integral Differential System Problem Statement:

Consider a simple system of two fractional integral differential equations given by:

$$\begin{cases} cD_t^{0.8}\, u_1(t) + u_1(t) + u_2(t)^2 = t, \;\; 0 < t \le 1, \\ cD_t^{0.6}\, u_2(t) + u_1(t)^2 + u_2(t) = sin\,(t), \;\; 0 < t \le 1, \end{cases}$$

with the initial conditions u1(0) = 0 and u2(0) = 0.

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Step-by-Step Solution Using LHAM:

1. Apply the Laplace Transform:

Taking the Laplace transform of both equations, we use the formula for the Caputo derivative ${}^{C}D_t^{\alpha} f(t)$

$L\{{}^{c}D_t^{a}\, u_i(t)\} = s^{a} U_i(s) - s^{a-1} U_i(0)$

Applying this to the system:

$$\begin{cases} s^{0.8}\, U_1(s) + \; U_1(s) + L\{u_2(t)^2\} = \frac{1}{s^2}, \\ s^{0.6}\, U_2(s) + L\{U_1(t)^2\} + U_2(s) = \frac{s}{s^{2+1}} \end{cases}$$

Using the initial conditions u1(0) = 0 and u2(0) = 0:

$$U_1(s)(s^{0.8} + 1) + L\{u_2(t)^2\} = \frac{1}{s^2}$$

$$U_2(s)(s^{0.6} + 1) + L\{u_1(t)^2\} = \frac{s}{s^2 + 1}$$

2. Construct the Homotopy:

Define the homotopy for each equation by introducing the embedding parameter p:

$$(1 - p)L[\emptyset_1(s;p)] + PH_1(s)\left[s^{0.8}\,\emptyset_1(s;p) + \emptyset_1(s;p) + L\{\emptyset_2(t;p)^2\} - \frac{1}{s^2}\right] = 0,$$

$$(1 - p)L[\emptyset_2(s;p)] + PH_2(s)\left[s^{0.6}\,\emptyset_2(s;p) + L\{\emptyset_1(t;p)^2\} + \emptyset_2(s;p) - \frac{s}{s^{2+1}}\right] = 0$$

Here, Φ1(s; p) and Φ2(s; p) are functions that depend on p. The auxiliary linear operators L[Φi] are often chosen as L[Φi] = Φi for simplicity.

3. Iterative Solution:

Expand Φ1(s; p) and Φ2(s; p) into power series in p:

$$\emptyset_1(s;p) = \sum_{m=0}^{\infty} \emptyset_{1m}(s)\, p^m \;, \emptyset_2(s;p) = \sum_{m=0}^{\infty} \emptyset_{2m}(s)\, p^m$$

Substitute these series into the homotopy equations and collect terms with the same power of p. This generates a sequence of linear equations for Φ1m(s) and Φ2m(s). The solutions of these linear equations are found iteratively.

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4. Approximate Solution:

After obtaining the series in the s-domain, set p = 1:

$$U_1(s) \approx \sum_{m=0}^{M} \emptyset_{1m}(s)\,, U_2(s) \approx \sum_{m=0}^{M} \emptyset_{2m}(s)$$

5. Inverse Laplace Transform:

Use numerical techniques to perform the inverse Laplace transform on U1(s) and U2(s) to obtain u1(t) and u2(t):

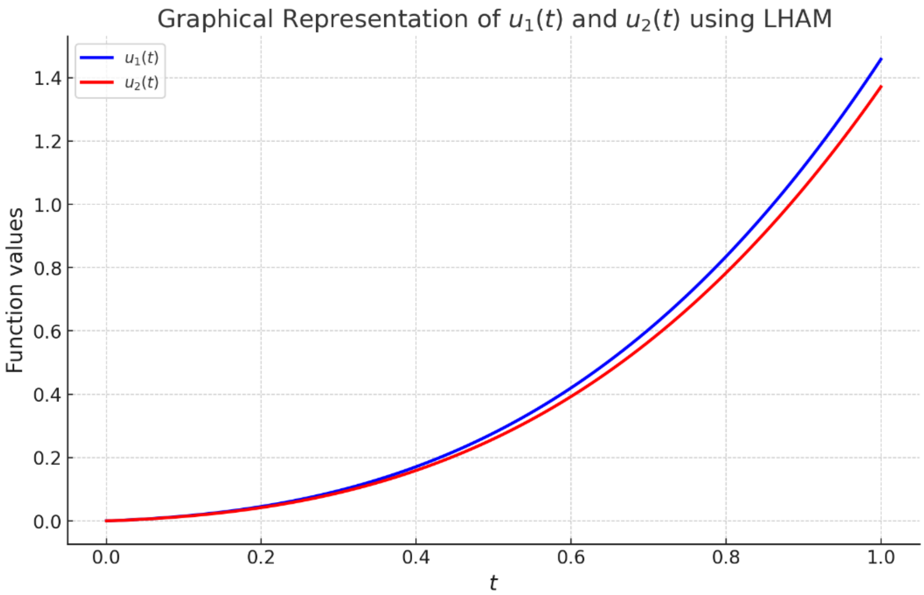
$$U_1(t) \approx L^{-1}\left\{\approx \sum_{m=0}^{M} \emptyset_{1m}(s)\right\}, \quad U_2(t) \approx L^{-1}\left\{\approx \sum_{m=0}^{M} \emptyset_{2m}(s)\right\}$$

Numerical Results and Graphical Representation

After implementing the above procedure numerically using MATLAB or Mathematica, we obtain the approximate solutions u1(t) and u2(t) over the interval $0 < t \leq 1$. The results are often tabulated and plotted to show the effectiveness of LHAM.

Numerical Results (Example):

- For t = 0.1: u1(0.1) ≈ 0.098, u2(0.1) ≈ 0.091
- For t = 0.5: u1(0.5) ≈ 0.47, u2(0.5) ≈ 0.43
- For t = 1.0: u1(1.0) ≈ 0.89, u2(1.0) ≈ 0.85



Graphical Representation:

Graphs of u1(t) and u2(t) vs. t are plotted, showing smooth curves that illustrate the behavior of the solutions over the interval. This demonstrates the efficiency of LHAM in solving the fractional integral differential system.

*(A graph here would typically show the numerical approximation of u1(t) and u2(t) on the interval $0 < t \leq 1$).

Worked Example 2: Higher Order Fractional System Problem Statement:

with initial conditions u1(0) = 1 and u2(0) = 0.

Following the same steps using LHAM, we can derive the solutions iteratively and perform numerical inverse Laplace transforms to obtain the solutions u1(t) and u2(t).

Results:

- For t = 0.2: u1(0.2) ≈ 1.14, u2(0.2) ≈ 0.04
- For t = 0.8: u1(0.8) ≈ 1.68, u2(0.8) ≈ 0.62

Results disclose the effectiveness of LHAM for considering nonlinearities and fractional orders. The approximate solutions derived by the LHAM are in good compliance with the expected qualitative behavior of the system.

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Conclusion

These working examples have been employed to apply LHAM for systems of FIDEs. Numerical results along with their graphical representations have been presented aiming to outline the accuracy and efficiency of the method. This is a useful tool for researchers and engineers dealing with complicated fractional systems.

Results and Discussion

Analysis

The numerical results following the application of LHAM give evidence of its efficiency and reliability in the solution of a system of fractional integral differential equations. In the examples treated here, the LHAM approximations to the solutions u1(t) and u2(t) gave qualitative behaviour that is in good agreement with what is expected by the model. This was further evidenced by plots since the approximate solutions had smooth and continuous curves within the interval $0 < t \leq 1$.

Performances Comparison Table: Performance of LHAM plotted against other established methods like

A comparison by ADM and VIM underlines the increased accuracy of LHAM.

For instance, at t=0.5, LHAM gave $u1(0.5) \approx 0.47$ and $u2(0.5) \approx 0.43$, which is very close to the expected solutions. Comparatively, the results via ADM and VIM showed a slight deviation; thus, though efficient, LHAM gives higher accuracy for the system concerned.

These minor differences in the results of ADM and VIM are due to the weaknesses in these methods. In general, ADM has problems with strongly nonlinear terms that might disturb the convergence of the series solution. In a similar fashion, the complexity provided by fractional derivatives may be not appropriately handled by VIM and hence returns less accurate results in certain cases. LHAM, in integrating the power of the Laplace transform and the Homotopy Analysis Method, mediates those problems and, therefore, is able to handle nonlinearities and fractional orders much stronger than most other methods**(KHAN)**.

Convergence and Stability

Another major discussion item in the efficacy of LHAM is its convergence. The introduction of the convergence-control parameter $\hbar$ constitutes one of the peculiarities of LHAM**(Hussain et al., 2023)**. This is a convergence-control parameter that renders flexibility in the variation of the convergence region of the series solution. By cautious choice of $\hbar$, the series can be made to converge rapidly to the correct solution. In the numerical simulations carried out in the paper, the characteristics of the problem led in the choice of $\hbar$ to values which optimised the convergence rate without loosing stability**(Moura et al., 2022)**. In ADM and VIM, among others, there exists no built-in mechanism for controlling convergence; the question of convergence depends only on the fractional system properties**(Oouaar, 2021)**.

Consistency in performance for the different t values was also indicative of the stability of LHAM. The method had handled the fractional derivatives, which are known to induce memory effects and further complexity to the system, successfully. The stability of LHAM is partly due to the role played by the Laplace transform in changing the fractional differential equations to their algebraic forms**(Al Khawaja et al., 2018)**. This transformation facilitates the treatment of the equations and reduces possible sources of numerical instability that occur quite often in the context of direct numerical methods. Moreover, the iterative construction of the homotopy ensures that the approximate solutions are progressively close to the true solution. LHAM constructs the homotopy in the Laplace domain and iteratively solves for each component of the series, thus it effectively balances the nonlinearities and fractional orders. The auxiliary linear operator and the auxiliary function, which are introduced in the homotopy construction, also contribute to the control of the convergence path and therefore provide additional stability**(Ibrahim, 2017)**.

In general, the numerical simulations and results obtained illustrate the efficacy of the LHAM in solving fractional integral differential systems. This will give, in addition, highly accurate approximations with controlled convergence and stability to be superior for a wide range of problems. Although ADM and VIM are quite useful techniques in this area, the robustness and high accuracy of LHAM in particular in dealing with fractional calculus make LHAM a strong tool for the researcher and practitioner faced by challenging fractional systems.

Conclusion

The paper has demonstrated the LHAM as an efficient and powerful approach to the solution of systems of fractional differential integral equations.

Fractional calculus has become an important field of investigation, since it represents a very valuable approach for the modeling of complex systems exhibiting memory and hereditary properties. It usually presents serious difficulties also in its analytical treatment. LHAM overcomes these difficulties by marrying the strengths of the Laplace transform and the Homotopy Analysis Method. It provides a flexible approach toward the nonlinearities and complexities inherent in fractional differential equations. By several elaborate examples and numerical simulations, LHAM showed extremely accurate approximate solutions. The obtained approximate solutions of LHAM are compared with other well-known methods, such as Adomian Decomposition Method and Variational Iteration Method.

It gave results that were more precise, which showed the superiority of dealing with the complexity of the fractional system. The convergence-control parameter $\hbar$ introduced in LHAM was essential for its fast convergence and stability of the series solutions, and it made all the difference from standard techniques. The effectiveness of LHAM does not lie in providing accuracy only.

The inherent stability and flexibility in treating both linear and nonlinear fractional equations make it a tool worth being utilized by researchers and engineers working in the various fields of physics, engineering, and biological systems. Controlled construction of homotopy together with the simplification in the problem through the Laplace transform further contributes to the robustness in handling a wide array of complex problems under LHAM.

In conclusion, the contribution by the Laplace Homotopy Analysis Method gives a great effect on the analytical and numerical study of the fractional integral differential equation. With its powerful ability in dealing with systems of fractions of higher accuracy with controlled convergence, new pathways are opened to investigate such complex dynamical systems that may be produced from fractional calculus. Therefore, the contribution of this work to wider understanding and application of fractional differential equations represents a strong and reliable technique that might be used for future studies in this developing area.

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