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Lie Symmetries and Partial Differential Equations(PDEs)

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Abstract

There are many well-known techniques for obtaining exact solutions for differential equations, but many of them are simply special cases of a few powerful symmetry methods. Lie symmetry methods are techniques for finding exact solutions of a wide variety of differential equations. In this paper, we discuss the use of Lie symmetries on PDEs of order two in two independent variables and show how they can be used to transform the governing equation into another equation with one less independent variable. In addition, for our case study we consider the PDE(the bond-pricing equation) and solve the equation using Lie symmetry methods.

Keywords: *Lie symmetries, the group of transformations*, the *infinitesimals generator, partial differential equations, invariant.*

الملخص
هناك العديد من التقنيات المعروفة للحصول على حلول دقيقة للمعادلات التفاضلية ، ولكن الكثير منهم مجرد حالات خاصة من عدد قليل من طرق التناظر القوية. تناظرات لي هي تقنيات لإيجاد الحلول الدقيقة لمجموعة واسعة من المعادلات التفاضلية. في هذا البحث نناقش استخدام تناظر لي للمعادلات التفاضلية الجزئية من الدرجة الثانية في اثنين من المتغيرات المستقلة و إظهار الكيفية التي يمكن استخدامها لتحويل المعادلة التفاضلية الجزئية المعطاة الى معادلة اخرى في متغير مستقل واحد. بالاضافة الى ذلك لقد ناقشنا المعادلة التفاضلية الجزئية لسندات التسعير وحلها باستخدام طرق لي للتناظر.

1. Introduction

We will start by considering the second order PDE [Bluman et al.][1]

$$\Delta(x,t,u,u_t,u_x,u_{xt},u_{xx},u_{tt}) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

To find the one- parameter group of transformations[lie][4]

$$x_1 = f(x,t,u,\epsilon) = x + \epsilon X(x,t,u) + O(\epsilon^2)$$

$$t_1 = g(x,t,u,\epsilon) = t + \epsilon T(x,t,u) + O(\epsilon^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$u_1 = h(x,t,u,\epsilon) = u + \epsilon U(x,t,u) + O(\epsilon^2)$$

that leaves the PDE invariant, we need to solve the condition [Goard][2]

$$\Gamma^{(2)}\Delta = 0 \,|_{\Delta=0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

where $\Gamma^{(2)}$ is the second extension (or prolongation) of the infinitesimal generator

$$\Gamma^{(1)} = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u}\,\cdot$$

We can write $\Gamma^{(2)}$ as

$$\Gamma^{(2)} = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + U_{[x]}\frac{\partial}{\partial u_x} + U_{[t]}\frac{\partial}{\partial u_t} + U_{[xx]}\frac{\partial}{\partial u_{xx}} + U_{[tt]}\frac{\partial}{\partial u_{tt}} + U_{[xt]}\frac{\partial}{\partial u_{xt}} \quad \ldots\ldots\ldots\ldots (4)$$

where $$U_{[i]} = D_i(U) - D_i(X)u_x - D_i(T)u_t$$

and $$U_{[ij]} = D_j\left(U_{[i]}\right) - D_j(X)u_{ix} - D_j(T)u_{it}$$

for i,j *being* x *or* $t\,\cdot$

This requirement $\Gamma^{(2)}\Delta = 0 \,|_{\Delta=0}$ leads to an overdetermined linear system of equations called the determining equations for the coefficients X, T, U. Then invariant solutions satisfy the invariant surface condition $Xu_x + Tu_t = U$,

which when solved for known functions X, T, U leads to the functional form of the solution

$$u = f\big(x, t, \varphi(z)\big), \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $z = z(x, t)$ and φ is arbitrary [Hill][3].

Substitution of this form into the governing equation $\Delta = 0$ leads to an equation for $\varphi(z)$ which is an ordinary differential equations.

2. The case study

When the short-term interest rate , r, follows the stochastic process

$$dr = u(r,t)dt + w(r,t)dX$$

where $dX \sim N(0, \sqrt{dt})$, *the price of a zero- coupon bond* $V(r,t)$, with expiry at $T = t$ satisfies

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where $\lambda(r,t)$is defined as the market price of risk [Wilmott et. al] [7].

We look to find one-parameter groups of transformations

$$r^* = r + \epsilon\rho(r,t,V) + O(\epsilon^2)$$

$$t^* = t + \epsilon\tau(r,t,V) + O(\epsilon^2) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

$$V^* = V + \epsilon\eta(r,t,V) + O(\epsilon^2)$$

so that the bond-pricing equation remains invariant [Olver][5].

Hence, we solve (3) $\Gamma^{(2)}\Delta = 0 \,|_{\Delta=0}$

where

$$\Delta = V_t + \frac{w^2}{2}V_{rr} + (u - \lambda w)V_r - rV = 0 \cdot$$
…………..………(8)

We consider the case

$$(u - \lambda w) = a - br^m \qquad \text{and} \qquad w = cr^n$$
……………...................(9)

where $a, b, c \neq 0 \cdot$

Substituting (9) in (8) we obtain

$$\Delta = V_t + \frac{c^2 r^{2n}}{2}V_{rr} + (a - br^m)V_r - rV = 0 \cdot$$

From (4)

$$\Gamma^{(2)} = \left(\rho\frac{\partial}{\partial r} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial V} + \eta_{[r]}\frac{\partial}{\partial V_r} + \eta_{[t]}\frac{\partial}{\partial V_t} + \eta_{[rr]}\frac{\partial}{\partial V_{rr}} + \eta_{[rt]}\frac{\partial}{\partial V_{rt}} + \eta_{[tt]}\frac{\partial}{\partial V_{tt}}\right) \dots \dots \dots \dots (10)$$

So for invariance we need to solve $\Gamma^{(2)}\Delta = 0 \mid_{\Delta=0}$

$$\Rightarrow \left(\rho\frac{\partial}{\partial r} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial V} + \eta_{[r]}\frac{\partial}{\partial V_r} + \eta_{[t]}\frac{\partial}{\partial V_t} + \eta_{[rr]}\frac{\partial}{\partial V_{rr}} + \eta_{[rt]}\frac{\partial}{\partial V_{rt}} + \eta_{[tt]}\frac{\partial}{\partial V_{tt}}\right)\left[V_t + \frac{c^2 r^{2n}}{2}V_{rr} + (a - br^m)V_r - rV\right] = 0 \cdot$$
………………………………………………(11)

Expanding (11) we get

$$\rho[(-mbr^{m-1}V_r) + nc^2 r^{2n-1}V_{rr} - V] + \eta[-r] + \eta_{[r]}(a - br^m) + \eta_{[t]} + \eta_{[rr]}\frac{c^2 r^{2n}}{2} = 0$$
subject to $\Delta=0$,

and where $\rho = \rho(r,t)$, $\tau = \tau(r,t)$, $\eta = \eta(r,t,V)$,

$$\eta_{[r]} = \eta_r + \eta_V V_r - \rho_r V_r - \tau_r V_t,$$
……………………………………….. (12. 1)

$\eta_{[t]} = \eta_t + \eta_V V_t - \tau_t V_t - \rho_t V_r$, ……………………………………….(12.2)

$\eta_{[rr]} = \eta_{rr} + 2\eta_{rV}\, V_r - \rho_{rr} V_r + \eta_V V_{rr} - 2\rho_r V_{rr} + \eta_{VV}\,(V_r)^2 - \tau_{rr} V_t - 2\tau_r V_{rt}$. … … … … . . (12.3)

By substituting equations (12. 1, 2, 3) in equation (11) and substituting Δ=0 into (11) and then equating coefficients of $V_{rt}, {V_r}^2, V_r, V_t, V_{rr}$ and 1 to 0, we get the following determining equations,

$-\tau_r\, c^2 r^{2n} = 0$ ……………………………………….. (a)

$\frac{c^2 r^{2n}}{2}\, \eta_{VV} = 0$ ……………………………………......(b)

$a\rho_r - br^m \rho_r + c^2 r^{2n} \eta_{rV} - \frac{c^2 r^{2n}}{2}\, \rho_{rr} - \rho_t + br^{m-1}\rho(-m+2) - \frac{2an}{r}\rho = 0$ … … … … . (c)

$\frac{-c^2 r^{2n}}{2}\, \tau_{rr} + 2\rho_r - \tau_t + \tau_r(br^m - a) - \frac{2n}{r}\rho = 0$ ……………………………………….(d)

$\frac{c^2 r^{2n}}{2} \eta_{rr} + \eta_t + (a - br^m)\eta_r + \eta_V rV - r\eta + (2n-1)\rho_V - 2\rho_r r\, V = 0$ ………………..(e)

From equation (a) we get

$$\tau_r = 0 \Rightarrow \tau = \tau(t).$$

From equation (b) we get $\eta_{VV} = 0 \Rightarrow \eta = f(r,t) + Vg(r,t)$

From equation (d) we obtain

$2\rho_r - \acute{\tau}(t) - \frac{2n}{r}\rho = 0 \Rightarrow 2\rho_r - \frac{2n}{r}\rho = \tau'(t)$

Solving this first order linear DE, we find the integrating factor

$R = e^{\int \frac{-n}{r} dr} = e^{-n \ln r} = r^{-n}$

so that

$$\frac{d}{dr}(r^{-n}\rho) = \frac{\tau'(t)}{2} r^{-n}$$

$$\Rightarrow \frac{\rho}{r^n} = \frac{\tau'(t)}{2} \frac{r^{1-n}}{1-n} + k, \quad n \neq 1$$

and

$$\frac{\rho}{r^n} = \frac{\tau'(t)}{2} \ln r + k_1, \quad n = 1$$

So we have

$$\rho = \frac{\tau'(t)}{2} \frac{r}{(1-n)} + kr^n, \quad n \neq 1$$

and

$$\rho = \frac{\tau'(t)}{2} r \ln r + k_1 r, \quad n = 1.$$

From (c) we obtain

$$a \frac{\tau'(t)}{2(1-n)} + acnr^{n-1} - br^m \left(\frac{\tau'(t)}{2(1-n)} + cnr^{n-1}\right) + c^2 r^{2n} f_r - \frac{c^2 r^{2n}}{2} . cn(n-1) r^{n-2} - \frac{\tau''(t)}{2} \frac{r}{(1-n)} + (2-m) br^{m-1} \left[\frac{r\tau'(t)}{2(1-n)} + cr^n\right] - \frac{2an}{r} \left[\frac{\tau'(t)}{2} \frac{r}{(1-n)} + cr^n\right] = 0 \qquad \text{.............................(13)}$$

In (13) we could equate coefficients of r assuming that the different functions of r are linearly independent. However we must also consider the cases where the functions of r are linearly dependent to help us find all the necessary cases to consider. We use the automated computer package *DIMSYM* [Sherring][6]. Using *DIMSYM* we find that for all m, n, a, b, c the PDE (8) with (9) has symmetries $\frac{\partial}{\partial t}$, $V\frac{\partial}{\partial V}$, $\varphi(r,t)\frac{\partial}{\partial V}$, where φ is any solution to the governing equation. However, for special choices of m, n, the PDE has extra symmetries that we find it in the following special cases:

3.SPECIAL CASES

1.When $n = m = 0$ we obtain the extra symmetry generators

$$\Gamma_1 = \frac{\partial}{\partial V}\left[\frac{t^2}{2} - \frac{ta}{c^2} + \frac{tb}{c^2} + \frac{r}{c^2}\right]V + \frac{\partial}{\partial r}t$$

$$\Gamma_2 = Vt\frac{\partial}{\partial V} + \frac{\partial}{\partial r}$$

$$\Gamma_3 = \frac{\partial}{\partial V}[t^4c^4 - 4t^3c^2a + 4t^3c^2b + 12t^2rc^2 + 4t^2a^2 - 8t^2ab + 4t^2b^2 - 8tra + 8trb - 4tc^2 + 4r^2]\frac{V}{8c^2} + \frac{\partial}{\partial t}t^2 + \frac{\partial}{\partial r}(\frac{t^3c^2}{2} + rt)$$

$$\Gamma_4 = \frac{\partial}{\partial V}[t^3c^4 - 3t^2c^2a + 3t^2c^2b + 6trc^2 + 2ta^2 - 4tab + 2tb^2 - 2ra + 2rb]\frac{V}{4c^2} + \frac{\partial}{\partial t}t + \frac{\partial}{\partial r}(\frac{3}{4}t^2c^2 + \frac{r}{2})$$

2. when $n = \frac{1}{2}$, $m = 0$ we obtain the extra symmetry generator

$$\Gamma_1 = [\tau''(t)r - \tau'a + \tau'b]\frac{V}{c^2}\frac{\partial}{\partial V} + \tau(t)\frac{\partial}{\partial t} + \tau'(t)r\frac{\partial}{\partial r}$$

where

$$\tau(t) = k_1 + k_2e^{\sqrt{2}ct} + k_3e^{-\sqrt{2}ct}$$

3. $n = \frac{1}{2}$, $m = 1$, we obtain the extra symmetry generator

$$\Gamma_1 = [-\tau(t)ab + \tau''(t)r + \tau'(t)rb - \tau'(t)a]\frac{V}{c^2}\frac{\partial}{\partial V} + \tau(t)\frac{\partial}{\partial t} + r\tau'(t)\frac{\partial}{\partial r}$$

where

$$\tau(t) = k_1 + k_2e^{\sqrt{b^2+2c^2}\,t} + k_3e^{-\sqrt{b^2+2c^2}\,t}$$

4.Example:

When $n = \frac{1}{2}$, $m = 0$, the governing equation

$$V_t + \frac{c^2 r}{2} V_{rr} + (a-b)V_r - rV = 0$$

has the symmetry with generator

$$\Gamma_1 = [\tau"(t)r - \tau' a + \tau' b]\frac{V}{c^2}\frac{\partial}{\partial V} + \tau(t)\frac{\partial}{\partial t} + \tau'(t) r \frac{\partial}{\partial r}$$

where $\tau(t) = A + Be^{\sqrt{2}ct} + De^{-\sqrt{2}ct}$

we let $a - b = k$.

Solving the corresponding invariant surface condition

$$\tau'(t) r \frac{\partial V}{\partial r} + \tau(t)\frac{\partial V}{\partial t} = \frac{V(\tau"(t)r - \tau' k)}{c^2}$$

We obtain

$$V = \varphi(\xi)\, e^{\frac{r\tau'}{c^2 \tau}}\, \tau^{\frac{-k}{c^2}} \qquad ;\xi = \frac{r}{\tau(t)}$$

Substitution of this functional form into the governing equation implies that φ needs to satisfy

$$c^2\, \xi\, \varphi" + 2\,k\,\varphi' + (8k_2k_3 - 2k_1^2)\,\xi\,\varphi = 0\cdot$$

Using Maple, we can solve this ODE for $\varphi(\xi)$ to get

$$\varphi(\xi) = A\,\xi^{\frac{1}{2}\frac{c^2-k^2}{c^2}}\, J_{\frac{-1}{2}+\frac{k}{c^2}}\left(\frac{\sqrt{8k_2k_3 - 2k_1^2}}{c}\xi\right) + B\xi^{\frac{1}{2}\frac{c^2-k^2}{c^2}}\, Y_{\frac{-1}{2}+\frac{k}{c^2}}\left(\frac{\sqrt{8k_2k_3 - 2k_1^2}}{c}\xi\right)$$

Then $V = \varphi(\xi)\, e^{\frac{r\tau'}{c^2\tau}}\, \tau^{\frac{-k}{c^2}}$, where $\xi = \frac{r}{\tau(t)}\cdot$

5. Conclusion

Symmetries are most useful in the study of differential equations and they are useful in different ways. For partial differential equations, Lie groups of transformation can lead to invariant solutions which result from solving a reduced equation with one less linear independent variable.

Secondly, a symmetry of a differential equation transforms solutions into other solutions and thus symmetries can be used to generate new solutions from old ones. This section explained the applications of Lie symmetries to partial differential equations by means of simple familiar examples.

Further, we also investigated the application of Lie symmetries to the PDE

$$V_t + \frac{w^2}{2} V_{rr} + (u - \lambda w) V_r - rV = 0$$

which models the price of zero-coupon bonds. For the coefficient cases considered when

$$w(r,t) = cr^n \qquad and \quad (u(r,t) - \lambda w(r,t)) = \mathrm{a} - \mathrm{b}r^m$$

we found that the equation admitted three simple symmetries but for special choices of constants n and $m,$ further symmetries could be found.

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