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Some Dynamical Properties of Rössler System

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الملخص
هذا البحث يتناول بعض الخواص الديناميكية لنظام روسلر. ندرس استقرارية النقاط الحرجة لنظام روسلر وكذلك استقرارية النظام عندما $b=0$. نبحث وجود تفرع متعدد الحرج وتفرع هوبف في نظام روسلر. حساب ثابت ليبونوف الأعظم عند النقاط الحرجة في حالات خاصة.

Abstract

This paper characterizes some dynamical properties of Rössler system. We consider the stability of equlibria points of Rössler system and also stability of system when $b=0$. We investigate existence of transcritical bifurcation and Hopf bifurcation in Rössler system. Compute the largest Lyaponuv exponent at the critical points in the special case.

Keywords: Rössler system, Nonlinear systems, Stability, Hopf bifurcation, Lyaponuv exponent.

1. Introduction

The science of nonlinear dynamics and chaos theory has sparked many researchers to develop mathematical models that simulate vector fields of nonlinear chaotic physical systems. Nonlinear phenomena arise in all fields of engineering, physics, chemistry, biology, economics, and sociology. Examples of nonlinear chaotic systems include planetary climate prediction models, neural network models, data compression, turbulence, nonlinear dynamical economics, information processing, preventing the collapse of power systems, high-performance circuits and devices, and liquid mixing with low power consumption (Lorenz, 1963 and Chen and Dong, 1998). The defining equations of the Rössler system are:

$$\dot{x} = -y - z,$$
$$\dot{y} = x + ay,$$
$$\dot{z} = b + z(x - c), \quad (1)$$

where $(x, y, z) \in R^3$ *and* a, b, c *with* $a \neq 0$ are the real parameters. Rössler studied the chaotic attractor with $a = b = 0.2$, *and* $c = 5.7$, though properties of $a = b = 0.1$ and $c = 5.7$ have been more commonly used since (Heinz et al, 2004).

Some properties of the Rössler system can be deduced via linear methods such as eigenvectors, but the main features of the system require non-linear methods such as Poincare maps and bifurcation diagrams. The original Rössler paper states the Rössler attractor was intended to behave similarly to the Lorenz attractor, but also be easier to analyze qualitatively. An orbit within the attractor follows an outward spiral close to the x, y plane around an unstable fixed point (Rossler, 1976).

Lyaponuv exponents measure the rate at which nearby orbits converge or diverge. There are as many Lyaponuv exponents as there are dimensions in the state space of the system, but the largest is usually the most important. Roughly speaking the (maximal) Lyaponuv exponent is the time constant, lambda, in the expression for the distance between two nearby orbits, Exp (lambda*t). If lambda is negative, then the orbits converge in time, and the dynamical system is insensitive to initial conditions. However, if lambda is positive, then the distance between nearby orbits grows exponentially in time, and the system exhibits sensitive dependence on initial conditions (Panaji, 2005). The paper is organized as follows: In the first section we given some results regarding the stability of equlibria. In section two we study the transcritical and Hopf bifurcations occurring at the equilibrium points in the general cases. In section three we calculate the largest Lyaponuv exponent for Rössler system at the equilibrium points in the special cases.

2. Stability analysis of the Rössler system

It is clear that if $c^2 > 4ab$ then Rössler system has two equilibriums points:

$$p_1\left(\frac{c-\sqrt{c^2-4ab}}{2}, \frac{-c+\sqrt{c^2-4ab}}{2a}, \frac{c-\sqrt{c^2-4ab}}{2a}\right)$$

$$p_2\left(\frac{c+\sqrt{c^2-4ab}}{2}, \frac{-c-\sqrt{c^2-4ab}}{2a}, \frac{c+\sqrt{c^2-4ab}}{2a}\right)$$

and if $c^2 < 4ab$ then Rössler system has not isolated equilibrium.

Theorem 1 (Wiggins**, 1990)**

The critical point $\bar{x}$ of the nonlinear vector field $\dot{x} = f(x), \quad x \in R^n$ is asymptotically stable if that all of the eigenvalues of Jacobian matrix $Df(\bar{x})$ have negative real parts.

Theorem 2

The following statements are true:

(i) The critical point $p_1\left(\frac{c-\sqrt{c^2-4ab}}{2}, \frac{-c+\sqrt{c^2-4ab}}{2a}, \frac{c-\sqrt{c^2-4ab}}{2a}\right)$ is asymptotically stable if $(a<0, b=c>0)$ *or* $(a=b<0, c>0)$ and otherwise it unstable critical point.

(ii) The critical point $p_2(\frac{c+\sqrt{c^2-4ab}}{2}, \frac{-c-\sqrt{c^2-4ab}}{2a}, \frac{c+\sqrt{c^2-4ab}}{2a})$ is an unstable critical point for all $a,b,c \neq 0$.

Proof: (i) the Jacobian matrix of Rössler system at the point p_1 is:

$$J(p_1) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c-\sqrt{c^2-4ab}}{2a} & 0 & \frac{-c-\sqrt{c^2-4ab}}{2} \end{bmatrix}.$$

The characteristic polynomial of $J(p_1)$ is:

$$\lambda^3 - \lambda^2(\frac{2a-c-\sqrt{c^2-4ab}}{2}) - \lambda(\frac{2a^2c-a^2\sqrt{c^2-4ab}-2a-c+\sqrt{c^2-4ab}}{2a}) + \sqrt{c^2-4ab} = 0. \quad (2)$$

The solutions of equation (2) depend on a,b,c in the following way:

1. $For\ (a>0, b>0, c>0)\ or\ (a>0, b>0, c<0)\ or\ (a>0, b<0, c>0)$
 $or\ (a>0, b<0, c<0)\ or\ (a<0, b<0, c>0)\ or\ (a<0, b>0, c<0)$

there are one negative real eigenvalues and two complex eigenvalues with positive real part.

2. $For\ (a<0, b=c>0)\ or\ (a=b<0, c>0)$ there are one negative real eigenvalues and two complex eigenvalues with negative real part.

(ii) the Jacobian matrix of Rössler system at the point p_2 is:

$$J(p_2) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c+\sqrt{c^2-4ab}}{2a} & 0 & \frac{-c+\sqrt{c^2-4ab}}{2} \end{bmatrix}.$$

The characteristic polynomial of $J(p_2)$ is:

$$\lambda^3-\lambda^2\left(\frac{2a-c+\sqrt{c^2-4ab}}{2}\right)-\lambda\left(\frac{a^2c-a^2\sqrt{c^2-4ab}-2a-c-\sqrt{c^2-4ab}}{2a}\right)-\sqrt{c^2-4ab}=0. \quad (3)$$

The solutions of equation (3) depend on a,b,c in the following way:

1. $For\ (a>0,b>0,c>0\ )\ or\ (a<0,b>c>0)\ or\ (a<0,b>0,c<0)$ there are one positive real eigenvalues and two complex eigenvalues with negative real part.

2. $For\ (a=b>0,c>0\ )\ or\ (a>0,b<0,c<0)$ there are three complex eigenvalues with positive real part.

3. $For\ (a<0,c>b>0)\ or\ (a<0,b<0,c<0)$ there are three complex eigenvalues one with positive real part and two with negative real part.

Next, we study Rössler system when $b=0$.

Rössler system becomes as follows:

$$\dot{x}=-y-z,$$
$$\dot{y}=x+ay,$$
$$\dot{z}=b+z(x-c), \quad (4)$$

with two parameters $a,c\ and\ a\neq 0$.

The system (4) have two equilibria points: $p^*{}_1(0,0,0)$, $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$.

Theorem 3

The following statements are true:

(i) If $(a<0\ and\ c>0)$ then the critical point $p^*{}_1(0,0,0)$ is asymptotically stable.

(ii) If $(a>0\ )or\ (c<0)$ then the critical point $p^*{}_1(0,0,0)$ is an unstable.

Proof: : (i) the Jacobian matrix of system (4) at the point $p^*_1(0,0,0)$ is:

$$J(p^*_1)=\begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}.$$

The characteristic polynomial of $J(p^*_1)$ is:

$$\lambda^3+(c-a)\lambda^2+(1-ac)\lambda+c=0$$

$$(\lambda+c)(\lambda^2-a\lambda+1)=0$$

$$\lambda_1=-c,\ \lambda_2=\frac{1}{2}(\ a-\sqrt{a^2-4}\ ),\ \lambda_3=\frac{1}{2}(\ a+\sqrt{a^2-4}\ )$$

It is clear if $c>0\ and\ -2\le a<0$ then $\lambda_1<0$, $\mathrm{Re}(\lambda_{2,3})<0$.

If $c>0\ and\ a<-2$ then $\lambda_{1,2,3}<0$.

Therefore, for $a<0\ and\ c>0$ the point $p^*_1(0,0,0)$ is asymptotically stable.

If $c<0$ then $\lambda_1>0$.If $0<a<2$ then $\mathrm{Re}(\lambda_{2,3})>0, if\ a>2\Rightarrow\lambda_3>0$.

Consequently, for $(a>0\ )\ or\ (c<0)$ the point $p^*_1(0,0,0)$ is an unstable.

Next, consider the stability of system (4) at the point $p^*_2(c,-\frac{c}{a},\frac{c}{a})$.

Under the transformation $(x,y,z)\to(X,Y,Z)$:

$$x=X+c$$

$$y=Y-\frac{c}{a}$$

$$z=Z+\frac{c}{a}$$

The system (4) becomes:

$$\dot{X} = -Y - Z$$
$$\dot{Y} = X + aY$$
$$\dot{Z} = XZ + \frac{c}{a}$$

(5)

Hence, one has to consider the stability of system (5) at $(0,0,0)$.

The Jacobian matrix of system (5) at the point $(0,0,0)$ is:

$$J(p^*{}_2) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c}{a} & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of J is:

$$\lambda^3 - a\lambda^2 + (\frac{a+c}{a})\lambda - c = 0.$$

(6)

Then, from Routh-Hurwits conditions, this equation has all roots with negative real parts if and only if $A > 0, C > 0$ and $AB - C > 0$ where

$A = -a$, $B = \dfrac{a+c}{a}$, $C = -c$, that is:

$$\left.\begin{matrix} a < 0 \\ c < 0 \end{matrix}\right\}$$

(7)

Consequently, we have the following theorem:

Theorem 4

The equilibrium point $p^*{}_2(c, -\frac{c}{a}, \frac{c}{a})$ is asymptotically stable if and only if $a < 0, c < 0$.

3. Transcritical and Hopf bifurcations

Consider the parameter c as bifurcation parameter.

(i) Bifurcation at the point $p^*{}_1(0,0,0)$:

Proposition 1

If $c=0$ the system (4) undergoes transcritical bifurcation at the point $p_1(0,0,0)$ for all $a<0$.

Proof: If $a<0, c<0$, we get $p^*{}_1(0,0,0)$ unstable point and $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$ stable. If $a<0, c>0$ we get $p^*{}_1(0,0,0)$ stable point and $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$ unstable. Therefore, the system (4) has transcritical bifurcation at $c=0$.

(ii) Bifurcation of the points $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$:

Assume that $a<0, c<0$, we have the coefficients of cubic polynomial (6) are all positive. Therefore, $f(\lambda)>0$ for all $\lambda>0$. Consequently there is instability $[\mathrm{Re}(\lambda)>0]$ only if there are two complex conjugate zeros for (6). Let these two zeros be $\lambda_1 = iw \text{ and } \lambda_2 = -iw$. Since $\lambda_1+\lambda_2+\lambda_3=-a$, we have $\lambda_3=-a$ which is stability for system (4). Then we have $f(\lambda_3)=-2a^3-a-2c$ and $c=c_0=\frac{-2a^3-a}{2}$. Thus, Hopf bifurcation may appear at the steady state $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$. According to theorem 4, the point $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$ losses stability when $c=c_0=\frac{-2a^3-a}{2}$.

Theorem 5

If $c = c_0 = \dfrac{-2a^3 - a}{2}$, then the system (4) undergoes a Hopf bifurcation at the equilibrium point $p^*{}_2(c, -\dfrac{c}{a}, \dfrac{c}{a})$.

Proof: If $c = c_0 = \dfrac{-2a^3 - a}{2}$, then the equation (6) becomes $\lambda^3 - a\lambda^2 + (\dfrac{2a^2+1}{2})\lambda + \dfrac{2a^3+a}{2} = 0$. Therefore, characteristic equation has a pair of purely imaginary roots $\lambda_{1,2} = \pm i\sqrt{\dfrac{2a^2+1}{2}}$ and a negative real root $\lambda_3 = -a$.

Differentiating both sides of equation (6) with respect to c, we obtain

$$3\lambda^2 \frac{d\lambda}{dc} - 2a\lambda\frac{d\lambda}{dc} + \frac{d\lambda}{dc} + \frac{c}{a}\frac{d\lambda}{dc} + \frac{1}{a}\lambda - 1 = 0$$

$$\frac{d\lambda}{dc} = \frac{a-\lambda}{3a\lambda^2 - 2a^2\lambda + a + c}$$

$$\lambda_c'(c_0) = \frac{a-\lambda}{3a\lambda^2 - 2a^2\lambda - a^3 + \frac{1}{2}a}, \; with\ \lambda = \pm i\sqrt{\frac{2a^2+1}{2}}$$

$$= \frac{a - i\sqrt{\frac{2a^2+1}{2}}}{2a^3 + 2a - 2ia^2\sqrt{\frac{2a^2+1}{2}}}$$

$$= \frac{1}{2}\frac{a - i\sqrt{\frac{2a^2+1}{2}}}{(a^3+a) - ia^2\sqrt{\frac{2a^2+1}{2}}}$$

$$=\frac{1}{2}\frac{(2a^4+\frac{3}{2}a^2)+i(-a\sqrt{\frac{2a^2+1}{2}})}{2a^6+\frac{5}{2}a^4+a^2}$$

Then, $\mathrm{Re}[\lambda_c'(c_0)]=\frac{1}{2}\frac{2a^4+\frac{3}{2}a^2}{2a^6+\frac{5}{2}a^4+a^2}>0$.

$$\mathrm{Im}[\lambda_c'(c_0)]=-\frac{1}{2}\frac{a\sqrt{\frac{2a^2+1}{2}}}{2a^6+\frac{5}{2}a^4+a^2}<0.$$

Therefore, $\lambda_c'(c_0)\neq 0$. According to Hopf bifurcation theorem in [1,2], the system (5) has display a Hopf bifurcation at $(0,0,0)$, so the system (4) display a Hopf bifurcation at the point $p^*{}_2(c,-\frac{c}{a},\frac{c}{a})$.

Theorem 6

If $c=c_0=\frac{-2a^3-a}{2}$ *and* $b=0$, then the Rössler system undergoes a Hopf bifurcation at the equilibrium point $p_2=p_2^*$.

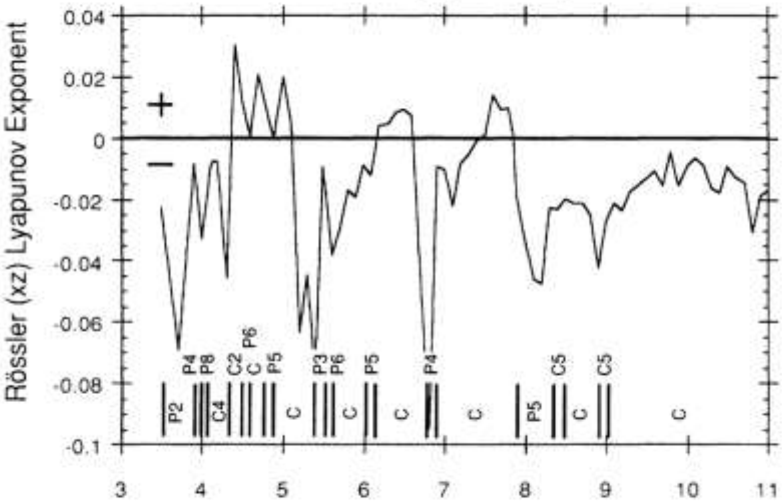
Proof: Directly from above theorem.

4. Lyaponuv exponent of Rössler system

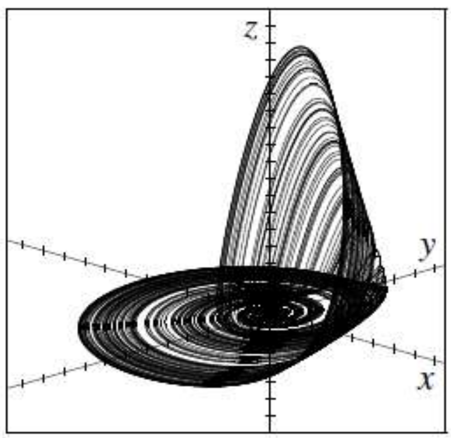
To compute the maximal Lyaponuv exponent of a system of ordinary differential equations we must integrate both the original system and its linearization $\dot{v}=A(t)v$. Essentially any initial vector v_0 can be used because almost all vectors will have some component along the direction of the maximal Lyaponuv direction. We cannot compute the limit in maximal Lyaponuv exponent $\gamma(x,v)=\lim_{t\to\infty}\sup\frac{1}{t}|\Phi(t,x)v|$ but instead simply integrate for some long time T and estimate: $\gamma_{\max}(T) = \frac{1}{T} Ln \frac{|v(T)|}{|v_0|}$.This quantity will rapidly converge to the maximal exponent; to estimate the error in the computation, it is useful to plot $\gamma_{\max}$ as a function of T (Meiss, 2007).

Next, we calculate maximal Lyaponuv exponent for Rössler system at the critical points by using Mat lab program.

1. The maximal Lyaponuv exponent for Rössler system at the critical point p_1 with parameters $(a<0, b=c>0)$ *and* $(a<b<0, c>0)$ *and* $(a<0, b=0, c>0)$ is negative number.
2. The maximal Lyaponuv exponent for Rössler system at the critical point p_1 with parameters $(a<0, b<0, c<0)$ *and* $(a>0, b>0, c>0)$ and $(a>0, b=0, c<0)$ and $(a<0, b=0, c<0)$ is positive number.
3. The maximal Lyaponuv exponent for Rössler system at the critical point p_2 with parameters $(a<0, b=0, c<0)$ is negative number
4. The maximal Lyaponuv exponent for Rössler system at the critical point p_2 with parameters $(a<0, b=c>0)$ *and* $(a<b<0, c>0)$ *and* $(a<0, b=0, c>0)$ is positive number.



Finger (1) For $3 < c < 11$, and constant parameters a and b there are three ranges for which the Rössler system has at least one nonnegative Lyapunov exponent, rendering convergence impossible.



Finger (2) The chaos for Rössler system when $a = 0.3$, $b = 0.4$ and $c = 0.5$

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